2012.3 Question 7

Since $\dot{y} = -2(y - z)$, differentiating both sides with respect to t gives

$$\begin{aligned}\ddot{y} &= -2\dot{y} + 2\dot{z} \\ &= -2\dot{y} + 2(-\dot{y} - 3z) \\ &= -4\dot{y} - 6z \\ &= -4\dot{y} - 3(\dot{y} + 2y) \\ &= -7\dot{y} - 6y,\end{aligned}$$

and hence

$$\ddot{y} + 7\dot{y} + 6y = 0.$$

The auxiliary equation

$$\lambda^2 + 7\lambda + 6 = 0$$

gives roots

$$\lambda_1 = -1, \lambda_2 = -6,$$

and hence

$$y = Ae^{-t} + Be^{-6t}.$$

Hence,

$$\dot{y} = -Ae^{-t} - 6Be^{-6t},$$

and therefore,

$$\begin{aligned}z &= \frac{\dot{y} + 2y}{2} \\ &= \frac{(-Ae^{-t} - 6Be^{-6t}) + 2(Ae^{-t} + Be^{-6t})}{2} \\ &= \frac{Ae^{-t} - 4Be^{-6t}}{2} \\ &= \frac{1}{2}Ae^{-t} - 2Be^{-6t}.\end{aligned}$$

This set of general solution

$$(y, z) = \left(Ae^{-t} + Be^{-6t}, \frac{1}{2}Ae^{-t} - 2Be^{-6t}\right),$$

is exactly what is desired.

1. $y(0) = 5$ and $z(0) = 0$ gives the system of linear equations

$$\begin{cases} A + B = 5, \\ \frac{1}{2}A - 2B = 0. \end{cases}$$

   This solves to $(A, B) = (4, 1)$. Hence,

$$z_1(t) = 2e^{-t} - 2e^{-6t}.$$

2. $z(0) = z(1) = c$ gives the system of linear equations

$$\begin{cases} \frac{1}{2}A - 2B = c, \\ \frac{1}{2e}A - \frac{2}{e^6}B = c, \end{cases} \implies \begin{cases} A - 4B = 2c, \\ e^5 A - 4B = 2e^6 c. \end{cases}$$

   Hence,

$$A = \frac{2c(e^6 - 1)}{e^5 - 1},$$

and therefore

$$\begin{aligned} B &= \frac{A - 2c}{4} \\ &= \frac{\frac{2c(e^6-1)}{e^5-1} - 2c}{4} \\ &= \frac{c}{2} \cdot \frac{(e^6 - 1) - (e^5 - 1)}{e^5 - 1} \\ &= \frac{ce^5(e-1)}{2(e^5-1)}. \end{aligned}$$

This gives

$$z_2(t) = \frac{c(e^6-1)}{e^5-1} e^{-t} - \frac{ce^5(e-1)}{e^5-1} e^{-6t}.$$

3. Notice that

$$\begin{aligned} &\sum_{n=-\infty}^{0} z_1(t-n) \\ &= \sum_{n=-\infty}^{0} [2e^{-t+n} - 2e^{-6t+6n}] \\ &= 2\sum_{n=0}^{\infty} [e^{-t-n} - e^{-6t-6n}] \\ &= 2\left[e^{-t}\sum_{n=0}^{\infty} e^{-n} - e^{-6t}\sum_{n=0}^{\infty} e^{-6n}\right] \\ &= 2\left[\frac{e^{-t}}{1-e^{-1}} - \frac{e^{-6t}}{1-e^{-6}}\right] \\ &= \frac{2e}{e-1}e^{-t} - \frac{2e^6}{e^6-1}e^{-6t}. \end{aligned}$$

Hence, c must be such that

$$\begin{cases} \frac{c(e^6-1)}{e^5-1} = \frac{2e}{e-1}, \\ \frac{2e^6}{e^6-1} = \frac{ce^5(e-1)}{e^5-1}. \end{cases}$$

Both solves to precisely

$$c = \frac{2e(e^5-1)}{(e-1)(e^6-1)},$$

and hence

$$z_2(t) = \sum_{n=-\infty}^{0} z_1(t-n)$$

for this value of c.