2012.3 Question 6

Since $x + iy$ is a root of this quadratic equation, putting it back into the original equation, we have

$$(x + iy)^2 + p(x + iy) + 1 = (x^2 - y^2 + px + 1) + (2x + p)yi = 0,$$

and so it must have both real parts and complex parts 0, and hence $x^2-y^2+px+1 = 0$, and $(2x+p)y = 0$.

Since $(2x + p)y = 0$, we must have either $2x + p = 0$ (which gives $p = -2x$), or $y = 0$. In the latter case, we put this back into the first equation, and we have

$$x^2 + px + 1 = 0.$$

If $x = 0$, then we must have $0+0+1 = 1 = 0$ which is impossible. Hence, $x \neq 0$, and by rearranging, we have

$$p = -\frac{x^2 + 1}{x}.$$

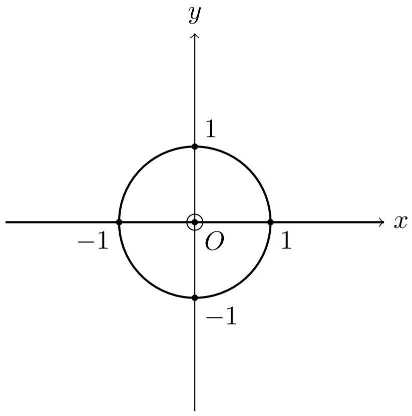
In the case where $p = -2x$, we must have

$$x^2 - y^2 + (-2x) \cdot x + 1 = 0 \iff x^2 + y^2 = 1,$$

and this represents a circle centred at the origin with radius 1.

In the case where $p = -\frac{x^2+1}{x}$, we must have $y = 0$, and $x \neq 0$. This represents the real axis without the origin.

This is the root locus of this equation.



For the second equation, let $z = x + iy$ be a solution. We have

$$p(x + iy)^2 + (x + iy) + 1 = (px^2 - py^2 + x + 1) + (2px + 1)yi = 0,$$

and so $px^2 - py^2 + x + 1 = 0$ and $(2px + 1)y = 0$.

Since $(2px+1)y = 0$, we must have either $2px+1 = 0$ (which gives $p = -\frac{1}{2x}$ since $x \neq 0$, or otherwise $0 + 1 = 1 = 0$), or $y = 0$. In the latter case, we put this back to the first equation, and we have

$$px^2 + x + 1 = 0.$$

If $x = 0$ then we must have $0+0+1 = 1 = 0$ which is impossible. Hence, $x \neq 0$, and by rearranging, we have

$$p = -\frac{x + 1}{x^2}.$$

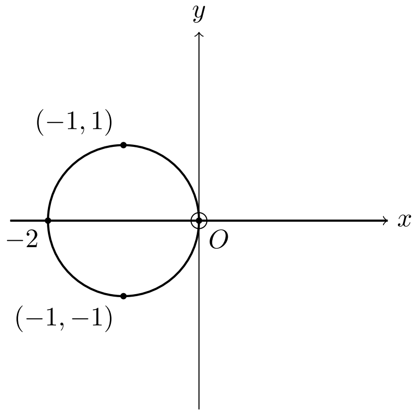
In the case where $p = -\frac{1}{2x}$, given $x \neq 0$,

$$-\frac{1}{2x}(x^2 - y^2) + x + 1 = 0 \iff \frac{x}{2} + \frac{y^2}{2x} + 1 = 0 \iff (x + 1)^2 + y^2 = 1.$$

This represents a circle centred at $(-1, 0)$ with radius 1, and since $x \neq 0$, we have to remove the point $(0, 0)$.

In the case where $p = -\frac{x+1}{x^2}$, $y = 0$ and this represents the real axis without the origin.

This is the root locus of this equation.



For the final equation, let $z = x + iy$ be a solution. We have

$$p(x+iy)^2 + p^2(x+iy) + 2 = (px^2 - py^2 + p^2x + 2) + yp(2x+p)i = 0,$$

and so $px^2 - py^2 + p^2x + 2 = 0$ and $yp(2x+p) = 0$.

Notice that here, $p \neq 0$, since if $p = 0$ then $2 = 0$ and there is no solution. So since $yp(2x+p) = 0$, we have $2x + p = 0$ which gives $p = -2x$, or $y = 0$. In the latter case, we put this back to the first equation, and we have

$$px^2 + p^2x + 2 = 0.$$

If $x = 0$ then we must have $0 + 0 + 2 = 2 = 0$ which is impossible. Hence, $x \neq 0$. For this to have a real solution for p, we must have $x \neq 0$ and

$$(x^2)^2 - 4 \cdot x \cdot 2 \geq 0,$$

which means

$$x(x-2)(x^2+2x+2) \geq 0.$$

Since $x^2 + 2x + 2 = (x+1)^2 + 1 \geq 1 \geq 0$, we must have $x(x-2) \geq 0$, and $x \leq 0$ or $x \geq 2$. This represents the real line with the interval $[0, 2)$ removed.

In the case where $p = -2x$, putting this back to the first equation, we have

$$(-2x)x^2 - (-2x)y^2 + (-2x)^2x + 2 = 0 \iff x^3 + xy^2 + 1 = 0 \iff y^2 = -\frac{1+x^3}{x}.$$

This is the root locus of this equation.