2012.3 Question 5

1. (a) An integer point: $(0, 1)$. A non-integer point: $\left(\frac{3}{5}, \frac{4}{5}\right)$.

   (b) An integer rational point: $(1, 1)$. Notice that

$$\begin{aligned}&(\cos\theta + \sqrt{m}\sin\theta)^2 + (\sin\theta - \sqrt{m}\cos\theta)^2 \\ &= \cos^2\theta + 2\sqrt{m}\sin\theta\cos\theta + m\sin^2\theta + \sin^2\theta - 2\sqrt{m}\sin\theta\cos\theta + m\cos^2\theta \\ &= (m+1)(\sin^2\theta + \cos^2\theta) \\ &= m+1.\end{aligned}$$

   Consider letting $x = \cos\theta + \sqrt{m}\sin\theta$, and $y = \sin\theta - \sqrt{m}\cos\theta$. Let $m = 1$, and we have $x = \cos\theta + \sin\theta$ and $y = \sin\theta - \cos\theta$, with $x^2 + y^2 = m + 1 = 2$.
   Let $\cos\theta = \frac{3}{5}$, and $\sin\theta = \frac{4}{5}$. We have

$$(x, y) = \left(\frac{7}{5}, \frac{1}{5}\right)$$

   is a non-integer rational point.

2. (a) An integer 2-rational point: $(1, \sqrt{2})$.
   For the non-integer 2-rational point, let $m = \sqrt{2}$ in the previous question, and we have

$$(\cos\theta + \sqrt{2}\sin\theta)^2 + (\sin\theta - \sqrt{2}\cos\theta)^2 = 2 + 1 = 3.$$

   Now, let $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$. Let $x = \cos\theta + \sqrt{2}\sin\theta = \frac{3}{5} + \sqrt{2}\cdot\frac{4}{5}$ and $y = \sin\theta - \sqrt{2}\cos\theta = \frac{4}{5} - \sqrt{2}\cdot\frac{3}{5}$. We must have $x^2 + y^2 = 3$, and

$$(x, y) = \left(\frac{3}{5} + \sqrt{2}\cdot\frac{4}{5}, \frac{4}{5} - \sqrt{2}\cdot\frac{3}{5}\right)$$

   is a non-integer 2-rational point on $x^2 + y^2 = 3$.

   (b) Consider $x = a\cos\theta + b\sqrt{m}\sin\theta$ and $y = a\sin\theta - b\sqrt{m}\cos\theta$, we have

$$\begin{aligned}x^2 + y^2 &= \left(a\cos\theta + b\sqrt{m}\sin\theta\right)^2 + \left(a\sin\theta - b\sqrt{m}\cos\theta\right)^2 \\ &= a^2\cos^2\theta + b^2m\sin^2\theta + 2ab\sqrt{m}\sin\theta\cos\theta \\ &\quad + a^2\sin^2\theta + b^2m\cos^2\theta - 2ab\sqrt{m}\sin\theta\cos\theta \\ &= (a^2 + b^2m)\cos^2\theta + (a^2 + b^2m)\sin^2\theta \\ &= (a^2 + b^2m)(\sin^2\theta + \cos^2\theta) \\ &= a^2 + b^2m.\end{aligned}$$

   We set $m = 2$, and hence we would like $a^2 + 2b^2 = 11$. Consider $a = 3$ and $b = 1$, and set $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$. Hence,

$$x = a\cos\theta + b\sqrt{m}\sin\theta = 3\cdot\frac{4}{5} + 1\cdot\sqrt{2}\cdot\frac{3}{5} = \frac{12}{5} + \sqrt{2}\cdot\frac{3}{5},$$

   and

$$y = a\sin\theta - b\sqrt{m}\cos\theta = 3\cdot\frac{3}{5} - 1\cdot\sqrt{2}\cdot\frac{4}{5} = \frac{9}{5} - \sqrt{2}\cdot\frac{4}{5},$$

   and we must have $x^2 + y^2 = 3^2 + 1^2\cdot 2 = 11$. Therefore,

$$(x, y) = \left(\frac{12}{5} + \sqrt{2}\cdot\frac{3}{5}, \frac{9}{5} - \sqrt{2}\cdot\frac{4}{5}\right)$$

   is a non-integer 2-rational point on the circle $x^2 + y^2 = 11$.

(c) Consider $x = a\sec\theta + b\sqrt{m}\tan\theta$ and $y = a\tan\theta + b\sqrt{m}\sec\theta$, we have

$$\begin{aligned}
x^2 - y^2 &= \left(a\sec\theta + b\sqrt{m}\tan\theta\right)^2 - \left(a\tan\theta + b\sqrt{m}\sec\theta\right)^2 \\
&= a^2\sec^2\theta + b^2m\tan^2\theta + 2ab\sqrt{m}\sec\theta\tan\theta \\
&\quad - a^2\tan^2\theta - b^2m\sec^2\theta - 2ab\sqrt{m}\sec\theta\tan\theta \\
&= a^2(\sec^2\theta - \tan^2\theta) - b^2m(\sec^2\theta - \tan^2\theta) \\
&= a^2 - b^2m.
\end{aligned}$$

We set $m = 2$, and hence we would like $a^2 - 2b^2 = 7$. Consider $a = 3$ and $b = 1$, and set $\tan\theta = \frac{3}{4}$ and $\sec\theta = \frac{5}{4}$. Hence,

$$x = a\sec\theta + b\sqrt{m}\tan\theta = 3\cdot\frac{5}{4} + 1\cdot\sqrt{2}\cdot\frac{3}{4} = \frac{15}{4} + \sqrt{2}\cdot\frac{3}{4},$$

and

$$y = a\tan\theta + b\sqrt{m}\sec\theta = 3\cdot\frac{3}{4} + 1\cdot\sqrt{2}\cdot\frac{5}{4} = \frac{9}{4} + \sqrt{2}\cdot\frac{5}{4},$$

and we must have $x^2 - y^2 = 3^2 - 1^2\cdot 2 = 7$. Therefore,

$$(x, y) = \left(\frac{15}{4} + \sqrt{2}\cdot\frac{3}{4}, \frac{9}{4} + \sqrt{2}\cdot\frac{5}{4}\right)$$

is a non-integer 2-rational point on the hyperbola $x^2 - y^2 = 7$.