2012.3 Question 4

1. Using the Maclaurin Expansion of e^x and setting $x = 1$, we have

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n+1}{n!} &= \sum_{n=1}^{\infty} \frac{n}{n!} + \sum_{n=1}^{\infty} \frac{1}{n!} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!} - \frac{1}{0!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} - 1 \\ &= e + e - 1 \\ &= 2e - 1.\end{aligned}$$

We have as well

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!} &= \sum_{n=1}^{\infty} \frac{n(n-1) + 3n + 1}{n!} \\ &= \sum_{n=1}^{\infty} \frac{n(n-1)}{n!} + 3\sum_{n=1}^{\infty} \frac{n}{n!} + \sum_{n=1}^{\infty} \frac{1}{n!} \\ &= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} + 3\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!} - 1 \\ &= 5\sum_{n=0}^{\infty} \frac{1}{n!} - 1 \\ &= 5e - 1,\end{aligned}$$

as desired.

We also have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(2n-1)^3}{n!} &= \sum_{n=1}^{\infty} \frac{8n^3 - 12n^2 + 6n - 1}{n!} \\ &= \sum_{n=1}^{\infty} \frac{8n(n-1)(n-2) + 12n(n-1) + 2n - 1}{n!} \\ &= 8\sum_{n=3}^{\infty} \frac{1}{(n-3)!} + 12\sum_{n=2}^{\infty} \frac{1}{(n-2)!} + 2\sum_{n=1}^{\infty} \frac{1}{(n-1)!} - \sum_{n=0}^{\infty} \frac{1}{n!} + 1 \\ &= (8 + 12 + 2 - 1)\sum_{n=0}^{\infty} \frac{1}{n!} + 1 \\ &= 21e + 1.\end{aligned}$$

2. Using the Maclaurin Expansion of $\ln(1-x)$ and letting $x = \frac{1}{2}$, we have

$$\ln 2 = -\ln\left(1 - \frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n} = \sum_{n=1}^{\infty} \frac{2^{-n}}{n}.$$

Hence,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(n^2+1)2^{-n}}{(n+1)(n+2)} &= \sum_{n=0}^{\infty} \frac{[(n+1)(n+2)-5(n+1)+2(n+2)]2^{-n}}{(n+1)(n+2)} \\ &= \sum_{n=0}^{\infty} 2^{-n} - 5\sum_{n=0}^{\infty} \frac{2^{-n}}{n+2} + 2\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1} \\ &= 2 - 5 \cdot 4 \sum_{n=2}^{\infty} \frac{2^{-n}}{n} + 2 \cdot 2 \sum_{n=1}^{\infty} \frac{2^{-n}}{n} \\ &= 2 - 20(\ln 2 - \frac{1}{2}) + 4(\ln 2) \\ &= -16 \ln 2 + 12.\end{aligned}$$