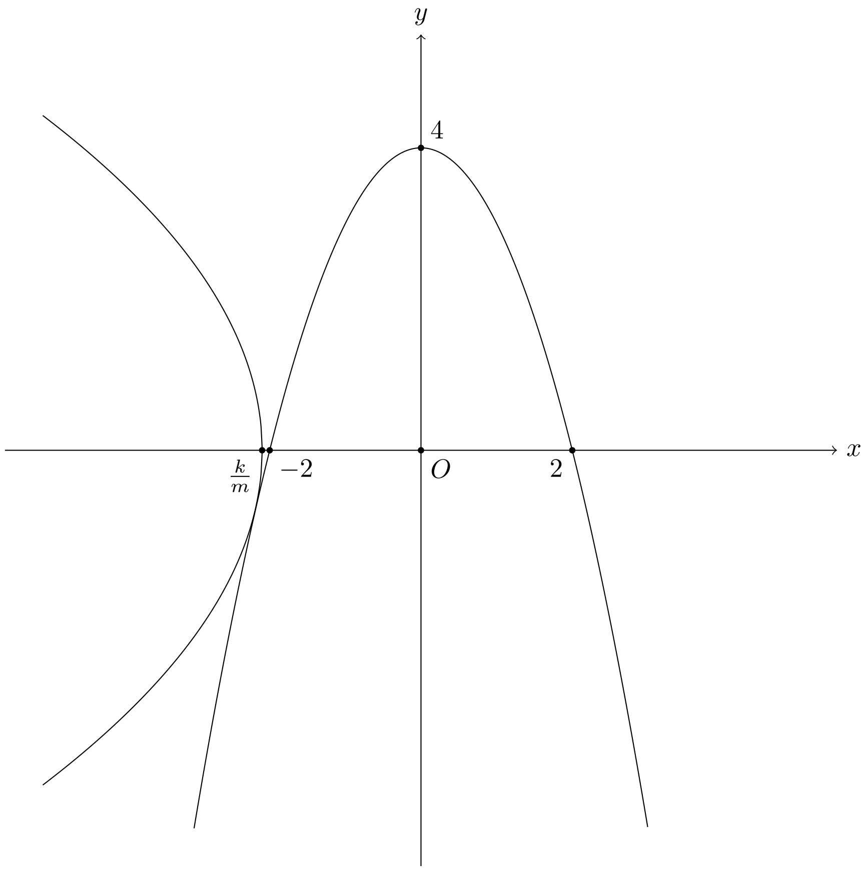
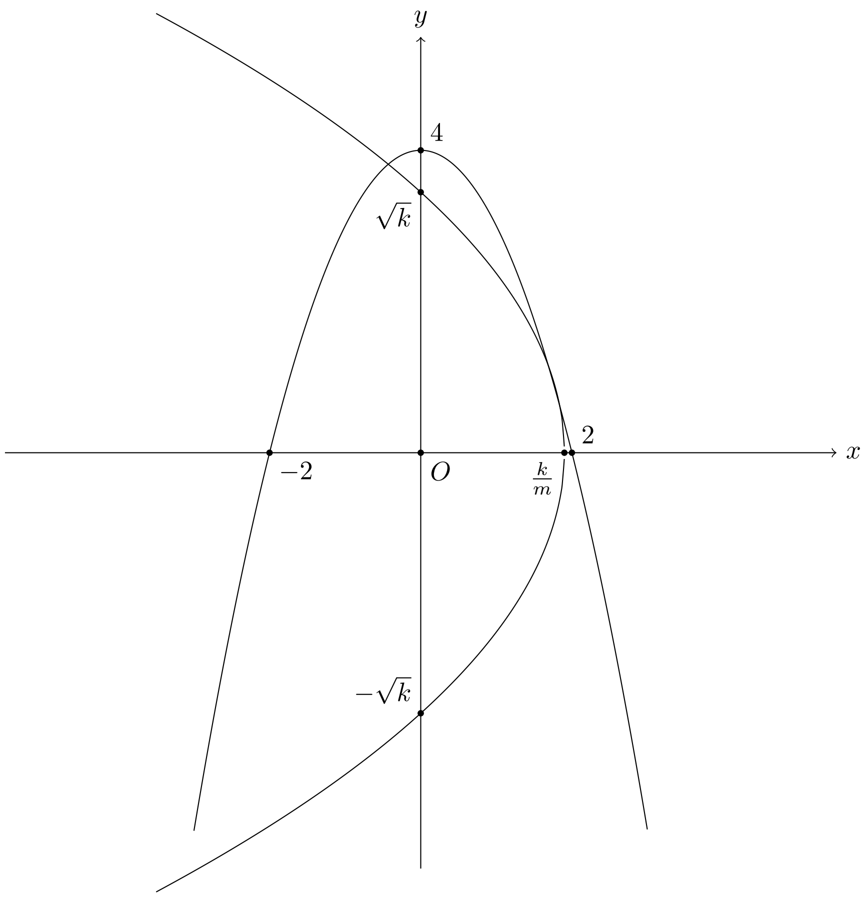
## 2012.3 Question 3

1. Let the two curves be $\Gamma_1 : y = 4 - x^2$ and $\Gamma_2 : x = -\frac{y^2}{m} + \frac{k}{m}$.
   For the first curve, its $y$-intercept is 4, and its $x$-intercept is $\pm 2$.
   For the second curve, its $y$-intercept is $\pm\sqrt{k}$ (if $k \geq 0$), and its $x$-intercept is $\frac{k}{m}$.

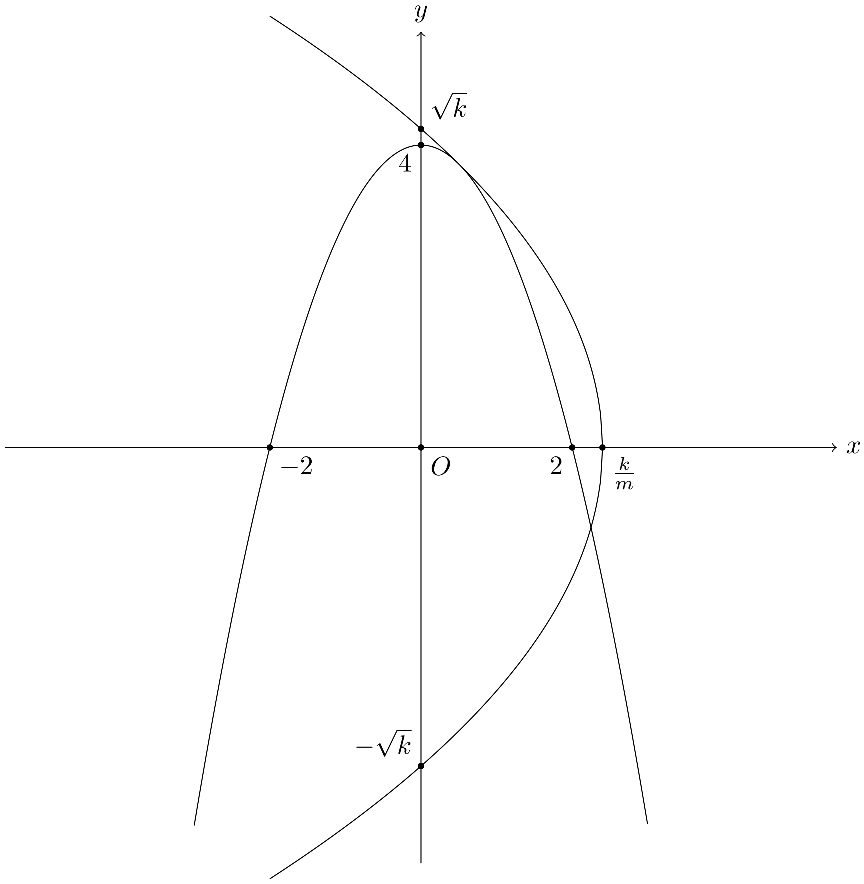
   (a) Since $k < 0$, we must have $\frac{k}{m} < 0$ as well, and hence the curves must look as follows:



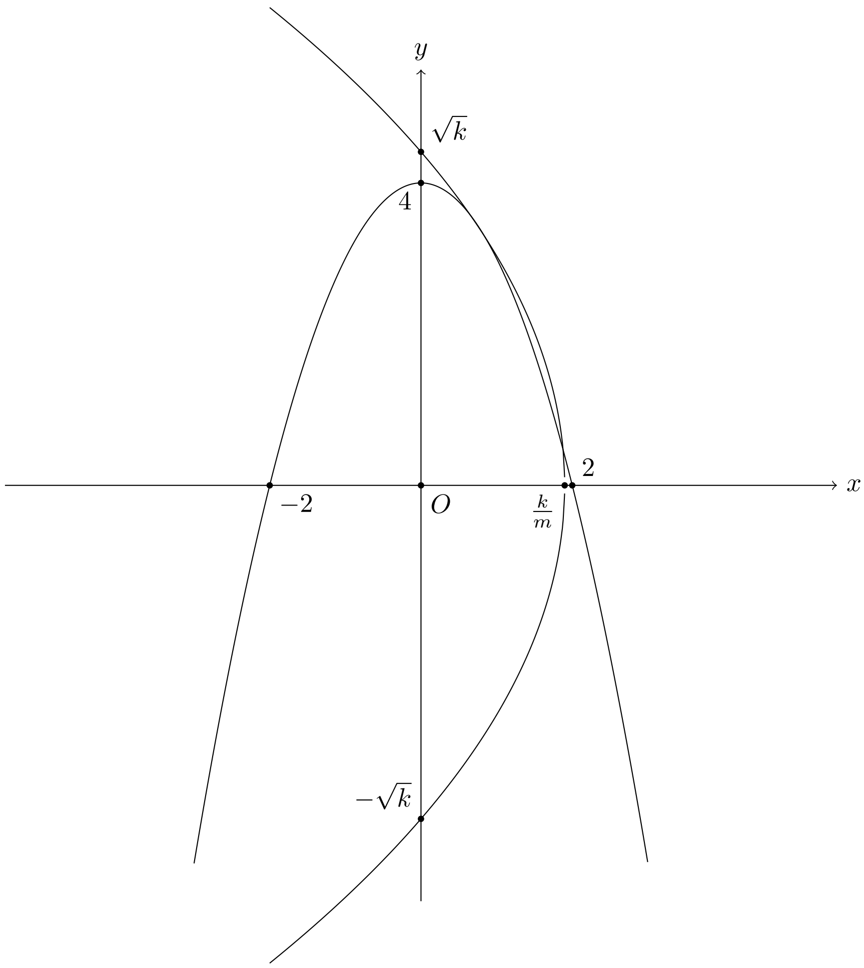

   (b) Since $0 < k < 16$, $\Gamma_2$ must have a $y$-intercept less than that of $\Gamma_1$. Since $\frac{k}{m} < 2$, $\Gamma_2$ must have the $x$-intercept to the left of $(2, 0)$. Hence, the curves must look as follows:




(c) Since $k > 16$, $\Gamma_2$ must have a $y$-intercept greater than that of $\Gamma_1$. Since $\frac{k}{m} > 2$, $\Gamma_2$ must have the $x$-intercept to the right of $(2, 0)$. Hence, the curves must look as follows:




(d) Since $k > 16$, $\Gamma_2$ must have a $y$-intercept greater than that of $\Gamma_1$. Since $\frac{k}{m} < 2$, $\Gamma_2$ must have the $x$-intercept to the left of $(2, 0)$. Hence, the curves must look as follows:




2. Since $y = y$, we must have

$$12x = k - (4 - x^2)^2 = k - 16 + 8x^2 - x^4,$$

and hence

$$x^4 - 8x^2 + 12x + 16 - k = 0,$$

as desired.

For the first curve, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x,$$

and applying implicit differentiation on both sides of the second equation, we must have

$$12 = -2y \frac{\mathrm{d}y}{\mathrm{d}x},$$

and hence

$$12 = (-2y) \cdot (-2x),$$

which gives $xy = 3$ for the point where the curves touch.

Hence,

$$\frac{3}{a} = 4 - a^2,$$

and this gives

$$a^3 - 4a + 3 = 0$$

as desired.

Notice that

$$a^3 - 4a + 3 = (a - 1)(a^2 + a - 3),$$

and hence the three solutions to $a$ are

$$a_1 = 1, a_{2,3} = \frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1 \pm \sqrt{13}}{2}.$$

From the first equation, we must have

$$\begin{aligned} k &= a^4 - 8a^2 + 12a + 16 \\ &= a(a^3 - 4a + 3) - 4a^2 + 9a + 16 \\ &= a \cdot 0 - 4a^2 + 9a + 16 \\ &= -4a^2 + 9a + 16, \end{aligned}$$

as desired.

For $a = 1$, $k = -4 \cdot 1^2 + 9 \cdot 1 + 16 = -4 + 9 + 16 = 21$, and $\frac{k}{m} = \frac{21}{12} < 2$, so (d) arises.

When $a_{2,3} = \frac{-1 \pm \sqrt{13}}{2}$, we have $a^2 + a - 3 = 0$, and hence

$$k = -4a^2 + 9a + 16 = -4(a^2 + a - 3) + 13a + 4 = 13a + 4.$$

For $a_2 = \frac{-1+\sqrt{13}}{2}$, we have

$$k = \frac{-13 + 13\sqrt{13}}{2} + 4 = \frac{-5 + 13\sqrt{13}}{2}.$$

Since $13\sqrt{13} > 13 \cdot 3 = 39$, we must have $-5 + 13\sqrt{13} > 34$, and hence $k > \frac{34}{2} = 17 > 16$.

We also have $13\sqrt{13} < 13 \cdot 4 = 52$, and hence $-5 + 13\sqrt{13} < 47$, and hence $k < \frac{47}{2}$, which means

$$\frac{k}{m} < \frac{47}{2 \cdot 12} = \frac{47}{24} < 2,$$

so case (d) arises.

For $a_3 = \frac{-1-\sqrt{13}}{2}$, we have $k = \frac{-13-13\sqrt{13}}{2} + 4 = \frac{-5-13\sqrt{13}}{4} < 0$, and so (a) arises.