2012.3 Question 2

1. By the formula for difference of two squares, we have

$$\begin{aligned}(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) &= (1-x^2)(1+x^2)(1+x^4)\cdots(1+x^{2^n})\\ &= (1-x^4)(1+x^4)\cdots(1+x^{2^n})\\ &= \cdots\\ &= 1-x^{2^{n+1}}.\end{aligned}$$

This means,

$$1=(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n})+x^{2^{n+1}},$$

and dividing both sides by $1-x$ gives

$$\frac{1}{1-x}=(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n})+\frac{x^{2^{n+1}}}{1-x}.$$

Rearranging and taking natural logs on both sides, we have

$$\ln(1-x^{2^{n+1}})-\ln(1-x)=\sum_{k=0}^{n}\ln(1+x^{2^k}),$$

and therefore,

$$\ln(1-x)=-\sum_{k=0}^{n}\ln(1+x^{2^k})+\ln(1-x^{2^{n+1}}).$$

Let $n\to\infty$. $2^{n+1}\to\infty$, and since $|x|<1$, we have $x^{2^{n+1}}\to 0$, and hence

$$\ln(1-x)=-\sum_{k=0}^{\infty}\ln(1+x^{2^k})+\ln(1)=-\sum_{k=0}^{\infty}\ln(1+x^{2^k}),$$

as desired.

Differentiating both sides with respect to x, we have

$$-\frac{1}{1-x}=-\sum_{k=0}^{\infty}\frac{2^k x^{2^k-1}}{1+x^{2^k}},$$

and hence

$$\frac{1}{1-x}=\sum_{k=0}^{\infty}\frac{2^k x^{2^k-1}}{1+x^{2^k}},$$

exactly as desired.

2. Notice that

$$\begin{aligned}&(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n})\\ &=((1+x^2)^2-x^2)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n})\\ &=(1+x^2+x^4)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n})\\ &=((1+x^4)^2-(x^2)^2)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n})\\ &=(1+x^4+x^8)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n})\\ &=\cdots\\ &=1+x^{2^n}+x^{2^{n+1}}.\end{aligned}$$

Therefore,

$$1=(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n})-x^{2^n}-x^{2^{n+1}},$$

and hence

$$\frac{1}{1+x+x^2} = (1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n}) - \frac{x^{2^n}+x^{2^{n+1}}}{1+x+x^2}.$$

Rearranging and taking natural logs on both sides, we have

$$\ln(1+x^{2^n}+x^{2^{n+1}}) - \ln(1+x+x^2) = \sum_{k=1}^{n} \ln(1-x^{2^{k-1}}+x^{2^k}),$$

and hence

$$\ln(1+x+x^2) = -\sum_{k=1}^{n} \ln(1-x^{2^{k-1}}+x^{2^k}) + \ln(1+x^{2^n}+x^{2^{n+1}}).$$

Let $n \to \infty$, we have $2^n, 2^{n+1} \to \infty$, and since $|x| < 1$, we must have $x^{2^n}, x^{2^{n+1}} \to \infty$, and hence $\ln(1+x^{2^n}+x^{2^{n+1}}) \to 0$. Hence,

$$\ln(1+x+x^2) = -\sum_{k=1}^{\infty} \ln(1-x^{2^{k-1}}+x^{2^k}).$$

Differentiating both sides with respect to x, we get

$$\frac{1+2x}{1+x+x^2} = -\sum_{k=1}^{\infty} \frac{-2^{k-1}x^{2^{k-1}-1}+2^k x^{2^k-1}}{1-x^{2^{k-1}}+x^{2^k}} = \sum_{k=1}^{\infty} \frac{2^{k-1}x^{2^{k-1}-1}-2^k x^{2^k-1}}{1-x^{2^{k-1}}+x^{2^k}},$$

which is exactly what is desired.