2012.3 Question 13

1. We have

$$\begin{aligned}
\mathrm{E}(Z \mid a < Z < b) &= \frac{\int_a^b z\Phi'(z)\,\mathrm{d}z}{\int_a^b \Phi'(z)\,\mathrm{d}z} \\
&= \frac{\int_a^b ze^{-\frac{z^2}{2}}\,\mathrm{d}z}{\sqrt{2\pi}\,(\Phi(b)-\Phi(a))} \\
&= \frac{\left[-e^{-\frac{z^2}{2}}\right]_a^b}{\sqrt{2\pi}\,(\Phi(b)-\Phi(a))} \\
&= \frac{e^{-\frac{a^2}{2}}-e^{-\frac{b^2}{2}}}{\sqrt{2\pi}\,(\Phi(b)-\Phi(a))}
\end{aligned}$$

2. Since $X = \mu + \sigma Z$

$$\begin{aligned}
\mathrm{E}(X \mid X > 0) &= \mathrm{E}(\mu+\sigma Z \mid (\mu+\sigma Z) > 0) \\
&= \mu + \sigma\,\mathrm{E}(Z \mid (\mu+\sigma Z) > 0) \\
&= \mu + \sigma\,\mathrm{E}\left(Z \mid Z > -\frac{\mu}{\sigma}\right),
\end{aligned}$$

as desired.

Hence,

$$\begin{aligned}
m &= \mathrm{E}(|X|) \\
&= \mathrm{E}(|X| \mid X > 0)\cdot \mathrm{P}(X>0) + \mathrm{E}(|X| \mid X < 0)\cdot \mathrm{P}(X<0) \\
&= \mathrm{E}(X \mid X > 0)\cdot \mathrm{P}(X>0) - \mathrm{E}(X \mid X < 0)\cdot \mathrm{P}(X<0) \\
&= \left[\mu + \sigma\,\mathrm{E}\left(Z \mid Z > -\frac{\mu}{\sigma}\right)\right]\cdot \mathrm{P}(\mu+\sigma Z>0) \\
&\quad - \left[\mu + \sigma\,\mathrm{E}\left(Z \mid Z < -\frac{\mu}{\sigma}\right)\right]\cdot \mathrm{P}(\mu+\sigma Z<0) \\
&= \left[\mu + \sigma\cdot\frac{\exp\left(-\frac{1}{2}\left(-\frac{\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}\left(1-\Phi\left(-\frac{\mu}{\sigma}\right)\right)}\right]\cdot\left[1-\Phi\left(-\frac{\mu}{\sigma}\right)\right] - \left[\mu + \sigma\cdot\frac{-\exp\left(-\frac{1}{2}\left(-\frac{\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}\Phi\left(-\frac{\mu}{\sigma}\right)}\right]\cdot\Phi\left(-\frac{\mu}{\sigma}\right) \\
&= \mu\left[1-\Phi\left(-\frac{\mu}{\sigma}\right)-\Phi\left(-\frac{\mu}{\sigma}\right)\right] + \frac{\sigma\exp\left(-\frac{1}{2}\left(-\frac{\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}}\cdot(1+1) \\
&= \mu\left[1-2\Phi\left(-\frac{\mu}{\sigma}\right)\right] + \frac{\sqrt{2}\sigma\exp\left(-\frac{1}{2}\cdot\frac{\mu^2}{\sigma^2}\right)}{\sqrt{\pi}},
\end{aligned}$$

as desired.

To find the variance of $|X|$, we would like to find $\mathrm{E}(|X|^2)$. But this is precisely $\mathrm{E}(|X|^2) = \mathrm{E}(X^2) = \mathrm{Var}(X) + \mathrm{E}(X)^2 = \sigma^2 + \mu^2$. Hence,

$$\begin{aligned}
\mathrm{Var}(|X|) &= \mathrm{E}(|X|^2) - \mathrm{E}(|X|)^2 \\
&= \sigma^2 + \mu^2 - m^2.
\end{aligned}$$