2012.3 Question 12

1. Let $[S]$ denote the area (2-D case) or the volume (3-D case) of S.

   Let $l = AB = BC = CA$, and hence we have

$$[\Delta ABC] = \frac{l \cdot 1}{2} = \frac{l}{2}.$$

   By trigonometry, we also have

$$[\Delta ABC] = \frac{l^2 \sin \frac{\pi}{3}}{2} = \frac{\sqrt{3}}{4} l^2,$$

   and hence

$$\frac{\sqrt{3}}{4} l^2 = \frac{l}{2} \iff l = \frac{2}{\sqrt{3}}.$$

   On the other hand, splitting up the triangle, we have

$$\begin{aligned}
[\Delta ABC] &= [\Delta ABP] + [\Delta BCP] + [\Delta ACP] \\
&= \frac{AB \cdot x_1}{2} + \frac{BC \cdot x_2}{2} + \frac{AC \cdot x_3}{2} \\
&= \frac{l}{2} (x_1 + x_2 + x_3).
\end{aligned}$$

   Since $[\Delta ABC] = [\Delta ABC]$, we must have $x_1 + x_2 + x_3 = 1$.

   Let the angle bisectors of $\angle BAC, \angle ABC$ and $\angle ACB$ meet at a point O (this point exists since triangle ABC is equilateral).

   For $X_1 = \min(X_1, X_2, X_3)$, this happens if and only if P is closer to AB than BC (including the equal case, $X_1 \leq X_2$), and P is closer to AB than AC (including the equal case, $X_1 \leq X_3$). This means P must lie on the side containing point A relative to BO (inclusive), and on the side containing point B relative to AO (inclusive).

   Hence, P must lie on or inside triangle AOB, as shown in the diagram below.

   Without loss of generality (since a triangle has order-3 rotational symmetry, and the centre of symmetry is O), we only consider the case where

$$X = X_1 = \min(X_1, X_2, X_3).$$

   This means P must lie on or inside triangle AOB. Consider the cumulative distribution function of X_1 under this condition. By the following diagram, for $0 \leq x \leq \frac{1}{3}$, we must have

$$\begin{aligned}
F(x) &= \mathrm{P}(X \leq x) \\
&\propto [\Delta ABO] - [\Delta ARQ] \\
&= \frac{l \cdot \frac{1}{3}}{2} \cdot \left[1 - \left(\frac{\frac{1}{3} - x}{\frac{1}{3}}\right)^2\right] \\
&= \frac{\frac{2}{\sqrt{3}} \cdot \frac{1}{3}}{2} \cdot \left[1 - (1 - 3x)^2\right] \\
&= \frac{1}{3\sqrt{3}} \cdot \left[6x - 9x^2\right] \\
&= \frac{2x - 3x^2}{\sqrt{3}}.
\end{aligned}$$

   The maximum of x is $\frac{1}{3}$, and hence $F\left(\frac{1}{3}\right) = 1$. This means the constant of proportionality, k, must satisfy

$$k = \frac{F\left(\frac{1}{3}\right)}{\left[\frac{2x-3x^2}{\sqrt{3}}\right]_{x=\frac{1}{3}}} = \frac{1}{\frac{1}{3\sqrt{3}}} = 3\sqrt{3},$$

and hence

$$F(x) = 3(2x - 3x^2).$$

Therefore, the probability density function of X for $0 \leq x \leq \frac{1}{3}$ must satisfy

$$f(x) = 6 - 18x = 6(1 - 3x),$$

and 0 everywhere else, i.e.

$$f(x) = \begin{cases} 6(1-3x), & 0 \leq x \leq \frac{1}{3}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the expectation of X satisfies

$$\begin{aligned}
\mathrm{E}(X) &= \int_{\mathbb{R}} x f(x) \,\mathrm{d}x \\
&= \int_0^{\frac{1}{3}} (6x - 18x^2) \,\mathrm{d}x \\
&= \left[3x^2 - 6x^3\right]_0^{\frac{1}{3}} \\
&= 3 \cdot \left(\frac{1}{3}\right)^2 - 6 \cdot \left(\frac{1}{3}\right)^3 \\
&= \frac{3}{9} - \frac{2}{9} \\
&= \frac{1}{9}.
\end{aligned}$$

2. Let the regular tetrahedron be $ABCD$ and the centroid be O. Let $AB = BC = BD = DA = l$. By trigonometry, we have

$$\frac{l^3}{6\sqrt{2}} = \frac{1}{3} \cdot \frac{\sqrt{3}l^2}{4} \cdot 1,$$

and hence

$$l = \frac{\sqrt{3}}{\sqrt{2}}.$$

Let the perpendicular distances from P to the face BCD, ACD, ABD and ABC be Y_1, Y_2, Y_3 and Y_4 respectively, and let

$$Y = \min(Y_1, Y_2, Y_3, Y_4).$$

By similar arguments as before, $Y_1 = \min(Y_1, Y_2, Y_3, Y_4)$ if and only if P is on or inside the tetrahedron $BCDO$.

Let G be the cumulative distribution function of Y_1 under this condition. For $0 \leq y \leq \frac{1}{4}$, we have

$$\begin{aligned}
G(y) &= \mathrm{P}(Y \leq y) \\
&\propto [BCDO] \cdot \left[1 - \left(\frac{\frac{1}{4} - y}{\frac{1}{4}}\right)^3\right] \\
&= \frac{1}{3} \cdot \frac{\sqrt{3}l^2}{4} \cdot \frac{1}{4} \cdot \left[1 - (1-4y)^3\right] \\
&= \frac{1}{16\sqrt{3}} \cdot \frac{3}{2} \cdot \left[12y - 48y^2 + 64y^3\right] \\
&= \frac{\sqrt{3}}{32} \cdot \left[12y - 48y^2 + 64y^3\right] \\
&= \frac{\sqrt{3}\left(3y - 12y^2 + 16y^3\right)}{8}.
\end{aligned}$$

Since the maximum of y is $\frac{1}{4}$, we must have $G\left(\frac{1}{4}\right) = 1$, and hence the constant of proportionality, k, must satisfy

$$k = \frac{G\left(\frac{1}{4}\right)}{\left[\frac{\sqrt{3}(3y-12y^2+16y^3)}{8}\right]_{y=\frac{1}{4}}} = \frac{1}{\frac{\sqrt{3}}{32}} = \frac{32}{\sqrt{3}}.$$

Hence,

$$G(y) = 4\left(3y - 12y^2 + 16y^3\right),$$

and the probability density function of Y must satisfy for $0 \leq y \leq \frac{1}{4}$

$$g(y) = 4\left(3 - 24y + 48y^2\right) = 12\left(1 - 8y + 16y^2\right).$$

Hence,

$$\begin{aligned}
\mathrm{E}(y) &= \int_{\mathbb{R}} y g(y) \, \mathrm{d}y \\
&= \int_0^{\frac{1}{4}} 12\left(y - 8y^2 + 16y^3\right) \mathrm{d}y \\
&= \left[6y^2 - 32y^3 + 48y^4\right]_0^{\frac{1}{4}} \\
&= 6 \cdot \left(\frac{1}{4}\right)^2 - 32 \cdot \left(\frac{1}{4}\right)^3 + 48 \cdot \left(\frac{1}{4}\right)^4 \\
&= \frac{3}{8} - \frac{1}{2} + \frac{3}{16} \\
&= \frac{3 \cdot 2 - 1 \cdot 8 + 3}{16} \\
&= \frac{1}{16}.
\end{aligned}$$