2012.3 Question 1

We have

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= n \cdot y^{n-1} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y^n \cdot 2 \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\ &= y^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}\left[n\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right],\end{aligned}$$

as desired.

1. Let $n = 1$, we have $z = y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, and

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right].$$

Hence, the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{y}$$

simplifies to

$$\frac{\frac{\mathrm{d}z}{\mathrm{d}x}}{\frac{\mathrm{d}y}{\mathrm{d}x}} = \sqrt{y},$$

and hence

$$\frac{\mathrm{d}z}{\mathrm{d}y} = \sqrt{y}.$$

Hence, by integration,

$$z = \frac{2}{3}y^{\frac{3}{2}} + C.$$

When $x = 0$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, and hence $z = 0$. Hence,

$$0 = \frac{2}{3} + C,$$

and therefore $C = -\frac{2}{3}$.
We therefore have

$$y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{2}{3}y^{\frac{3}{2}} - \frac{2}{3},$$

and hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{2}{3}\left(\sqrt{y} - \frac{1}{y}\right)}.$$

Rearrangement gives

$$\frac{\sqrt{y}\,\mathrm{d}y}{\sqrt{y^{\frac{3}{2}} - 1}} = \sqrt{\frac{2}{3}}\,\mathrm{d}x.$$

Notice that

$$\begin{aligned}\frac{\mathrm{d}\sqrt{y^{\frac{3}{2}} - 1}}{\mathrm{d}y} &= \frac{1}{2} \cdot \frac{1}{\sqrt{y^{\frac{3}{2}} - 1}} \cdot \frac{3}{2} \cdot \sqrt{y} \\ &= \frac{3}{4} \cdot \frac{\sqrt{y}}{\sqrt{y^{\frac{3}{2}} - 1}},\end{aligned}$$

and hence by integration

$$\frac{4}{3} \cdot \sqrt{y^{\frac{3}{2}} - 1} = \sqrt{\frac{2}{3}} x + C.$$

When $x = 0, y = 1$, and hence $C = 0$. Therefore,

$$\sqrt{y^{\frac{3}{2}} - 1} = \sqrt{\frac{3}{8}} x,$$

and hence

$$y^{\frac{3}{2}} = \frac{3}{8} x^2 + 1,$$

and hence

$$y = \left(\frac{3}{8} x^2 + 1\right)^{\frac{2}{3}},$$

as desired.

2. Let $n = -2$, we have $z = y^{-2} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, and

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -2y^{-3} \frac{\mathrm{d}y}{\mathrm{d}x} \left[\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right].$$

Hence, the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y^2 = 0$$

simplifies to

$$\frac{\frac{\mathrm{d}z}{\mathrm{d}x}}{-2y^{-3} \frac{\mathrm{d}y}{\mathrm{d}x}} + y^2 = 0,$$

which gives

$$\frac{\mathrm{d}z}{\mathrm{d}y} = \frac{2}{y}.$$

By integration on both sides, we have

$$z = 2 \ln y + C,$$

and when $x = 0$, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, which gives $z = 0$. Hence, $C = 0$, and

$$y^{-2} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 2 \ln y,$$

which gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \sqrt{2 \ln y},$$

and therefore,

$$\frac{\mathrm{d}y}{y \sqrt{\ln y}} = \sqrt{2}\, \mathrm{d}x.$$

By integration,

$$\int \frac{\mathrm{d}y}{y \sqrt{\ln y}} = \int \frac{\mathrm{d} \ln y}{\sqrt{\ln y}} = 2 \sqrt{\ln y} + C,$$

and hence

$$2 \sqrt{\ln y} = \sqrt{2} x + C.$$

When $x = 0$, $y = 1$, so $C = 0$, and hence

$$\sqrt{\ln y} = \frac{x}{\sqrt{2}},$$

and therefore, the solution to the original differential equation is

$$y = e^{\frac{x^2}{2}}.$$