2011.3 Question 8

Since $w = u + iv$, $z = x + iy$, we have

$$\begin{aligned} u + iv &= w \\ &= \frac{1+iz}{i+z} \\ &= \frac{1+i(x+iy)}{i+(x+iy)} \\ &= \frac{(1-y)+xi}{x+(y+1)i} \\ &= \frac{(1-y)+xi}{x+(y+1)i} \cdot \frac{x-(y+1)i}{x-(y+1)i} \\ &= \frac{[(1-y)+xi]\,[x-(y+1)i]}{x^2+(y+1)^2} \\ &= \frac{(1-y)x+x(y+1)}{x^2+(y+1)^2} + \frac{x^2-(1-y)\cdot(y+1)}{x^2+(y+1)^2} \cdot i \\ &= \frac{2x}{x^2+(y+1)^2} + \frac{x^2+y^2-1}{x^2+(y+1)^2} \cdot i, \end{aligned}$$

and hence

$$(u, v) = \left(\frac{2x}{x^2+(y+1)^2}, \frac{x^2+y^2-1}{x^2+(y+1)^2}\right).$$

1. When $y = 0$, we have

$$(u, v) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right).$$

   Let $x = \tan\left(\frac{\theta}{2}\right)$. The tangent half-angle substitution also gives that $u = \sin\theta$ and $v = -\cos\theta$, and hence $u^2 + v^2 = 1$.

   For the range of θ, we have $-\frac{\pi}{2} < \frac{\theta}{2} < \frac{\pi}{2}$, which means $-\pi < \theta < \pi$.

   This represents the unit circle without the point $(\sin\pi, -\cos\pi) = (0, 1)$ corresponding to $\theta = \pi(+2k\pi)$ for some integer k.

2. When $-1 < x < 1$, we have $-\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{4}$, which means $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. This is the unit circle with only the part below the u axis (exclusive).

3. When $x = 0$, we have

$$(u, v) = \left(0, \frac{y^2-1}{(y+1)^2}\right).$$

   Notice that

$$v = \frac{y^2-1}{(y+1)^2} = \frac{(y+1)(y-1)}{(y+1)^2} = \frac{y-1}{y+1} = 1 - \frac{2}{y+1},$$

   and hence $-1 < v < 1$.

   This means the locus of w is the line segment $u = 0, -1 < v < 1$.

4. When $y = 1$, we have

$$(u, v) = \left(\frac{2x}{x^2+4}, \frac{x^2}{x^2+4}\right).$$

   First, let $x = 2t$, and we have

$$(u, v) = \left(\frac{4t}{4t^2+4}, \frac{4t^2}{4t^2+4}\right) = \left(\frac{t}{t^2+1}, \frac{t^2}{t^2+1}\right).$$

   Let $t = \tan\left(\frac{\theta}{2}\right)$, and we have $-\pi < \theta < \pi$. Notice that

$$u = \frac{1}{2} \cdot \frac{2t}{t^2+1} = \frac{1}{2}\sin\theta,$$

and

$$v - \frac{1}{2} = \frac{1}{2} \cdot \frac{t^2 - 1}{t^2 + 1} = -\frac{1}{2} \cos \theta.$$

This means the loci is a subset of the circle centred at $\left(0, \frac{1}{2}\right)$ with radius $\frac{1}{2}$, with the point

$$(u, v) = \left(\frac{1}{2} \sin \pi, \frac{1}{2} - \frac{1}{2} \cos \pi\right) = (0, 1)$$

missing, which corresponds to $\theta = \pi(+2k\pi)$ for some integer k.