## 2011.3 Question 7

1. The base case is when $n = 2$, and we have

$$T_2 = (\sqrt{a+1} + \sqrt{a})^2 = (2a+1) + 2 \cdot \sqrt{a(a+1)}.$$

We therefore have $A_2 = 2a + 1$ and $B_2 = 2$, and we verify that

$$a(a+1)B_2^2 + 1 = a(a+1) \cdot 2^2 + 1 = 4a^2 + 4a + 1 = (2a+1)^2 = A_2^2,$$

as desired, and the statement holds for the base case when $n = 2$.

Now, assume that this statement is for some even $n = k$, i.e.

$$T_k = A_k + B_k \sqrt{a(a+1)}$$

where $A_k$ and $B_k$ are both integers, and $A_k^2 = a(a+1)B_k^2 + 1$.

Notice that

$$\begin{aligned}
T_{k+2} &= T_k \cdot \left(\sqrt{a+1} + \sqrt{a}\right)^2 \\
&= \left(A_k + B_k \sqrt{a(a+1)}\right) \cdot \left(2a + 1 + 2\sqrt{a(a+1)}\right) \\
&= A_k \cdot (2a+1) + B_k \cdot 2 \cdot a(a+1) + 2A_k \sqrt{a(a+1)} + (2a+1)B_k \sqrt{a(a+1)} \\
&= [(2a+1)A_k + 2a(a+1)B_k] + [2A_k + (2a+1)B_k] \sqrt{a(a+1)}.
\end{aligned}$$

Now let $A_{k+2} = (2a+1)A_k + 2a(a+1)B_k$, and $B_{k+2} = 2A_k + (2a+1)B_k$. Since $a$ is a positive integer, and $A_k$ and $B_k$ are both integers, we must have $A_{k+2}$ and $B_{k+2}$ are both integers. Furthermore,

$$\begin{aligned}
& A_{k+2}^2 - \left[a(a+1)B_{k+2}^2 + 1\right] \\
&= [(2a+1)A_k + 2a(a+1)B_k]^2 - \left[a(a+1)\left(2A_k + (2a+1)B_k\right)^2 + 1\right] \\
&= \left[(2a+1)^2 - 4a(a+1)\right] A_k^2 \\
&\quad + [2 \cdot (2a+1) \cdot 2a(a+1) - 2 \cdot a(a+1) \cdot 2 \cdot (2a+1)] A_k B_k \\
&\quad + [(2a(a+1))^2 - a(a+1)(2a+1)^2] B_k - 1 \\
&= A_k^2 - a(a+1)B_k^2 - 1 \\
&= 1 - 1 \\
&= 0,
\end{aligned}$$

and hence

$$A_{k+2}^2 = a(a+1)B_{k+2}^2 + 1.$$

So the original statement holds for $n = k + 2$.

By the principle of mathematical induction, the original statement must hold for all even integers $n$.

2. If $n$ is odd, then we have

$$\begin{aligned}
T_n &= (\sqrt{a+1} + \sqrt{a})T_{n-1} \\
&= (\sqrt{a+1} + \sqrt{a})(A_{n-1} + B_{n-1}\sqrt{a(a+1)}) \\
&= A_{n-1}\sqrt{a+1} + A_{n-1}\sqrt{a} + B_{n-1}(a+1)\sqrt{a} + B_{n-1}a\sqrt{a+1} \\
&= (A_{n-1} + aB_{n-1})\sqrt{a+1} + (A_{n-1} + (a+1)B_{n-1})\sqrt{a}.
\end{aligned}$$

Now, consider $C_n = A_{n-1} + aB_{n-1}$, and $D_n = A_{n-1} + (a+1)B_{n-1}$. Since $a$ is a positive integer, and $A_{n-1}$ and $B_{n-1}$ are integers, we must have $C_n$ and $D_n$ are integers as well. Furthermore,

$$\begin{aligned}
&(a+1)C_n^2 - (aD_n^2+1) \\
&= (a+1)\left(A_{n-1}+aB_{n-1}\right)^2 - \left[a\left(A_{n-1}+(a+1)B_{n-1}\right)^2+1\right] \\
&= \left[(a+1)-a\right]A_{n-1}^2 + \left[(a+1)\cdot 2\cdot a - a\cdot 2\cdot (a+1)\right]A_{n-1}B_{n-1} \\
&\quad + [(a+1)a^2 - a(a+1)^2]B_{n-1}^2 - 1 \\
&= A_{n-1}^2 - a(a+1)B_{n-1}^2 - 1 \\
&= 1-1 \\
&= 0,
\end{aligned}$$

and hence

$$(a+1)C_n^2 = aD_n^2 + 1.$$

This shows precisely the original statement.

3. For even $n$,

$$T_n = A_n + B_n\sqrt{a(a+1)} = \sqrt{A_n^2} + \sqrt{B_n^2\cdot a(a+1)} = \sqrt{A_n^2} + \sqrt{A_n^2-1},$$

and for odd $n$,

$$T_n = C_n\sqrt{a+1} + D_n\sqrt{a} = \sqrt{C_n^2(a+1)} + \sqrt{D_n^2 a} = \sqrt{aD_n^2+1} + \sqrt{aD_n^2},$$

as desired.