2011.3 Question 6

We show that T is equal to each of U, V, X, and by transitivity, this shows that all four are equal.

- To show $T = U$, consider the substitution $u = 2 \operatorname{artanh} t$, and hence $t = \tanh \frac{u}{2}$.

  When $t = \frac{1}{2}$, $u = 2 \operatorname{artanh} \frac{1}{2} = 2 \cdot \frac{1}{2} \cdot \ln \left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right) = \ln 3$, and when $t = \frac{1}{3}$, $u = 2 \operatorname{artanh} \frac{1}{3} = 2 \cdot \frac{1}{2} \cdot \ln \left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}} \right) = \ln 2$.

  We have $\mathrm{d}u = \frac{2}{1-t^2} \, \mathrm{d}t$, and hence

$$\begin{aligned} T &= \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t} \, \mathrm{d}t \\ &= \int_{\ln 2}^{\ln 3} \frac{\frac{u}{2}}{\tanh \frac{u}{2}} \cdot \frac{1 - \tanh^2 \frac{u}{2}}{2} \, \mathrm{d}u \\ &= \int_{\ln 2}^{\ln 3} \frac{u}{2} \cdot \frac{1 - \tanh^2 \frac{u}{2}}{2 \tanh \frac{u}{2}} \, \mathrm{d}u \\ &= \int_{\ln 2}^{\ln 3} \frac{u}{2 \sinh u} \, \mathrm{d}u \\ &= U. \end{aligned}$$

- To show $T = V$, we use integration by parts.

$$\begin{aligned} T &= \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t} \, \mathrm{d}t \\ &= \int_{\frac{1}{3}}^{\frac{1}{2}} \operatorname{artanh} t \, \mathrm{d} \ln t \\ &= [\operatorname{artanh} t \ln t]_{\frac{1}{3}}^{\frac{1}{2}} - \int_{\frac{1}{3}}^{\frac{1}{2}} \ln t \, \mathrm{d} \operatorname{artanh} t \\ &= \left(\operatorname{artanh} \frac{1}{2} \ln \frac{1}{2} - \operatorname{artanh} \frac{1}{3} \ln \frac{1}{3} \right) - \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln t}{1 - t^2} \, \mathrm{d}t \\ &= \left(\frac{1}{2} \cdot \ln \left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} \right) \cdot (-\ln 2) - \frac{1}{2} \cdot \ln \left(\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} \right) \cdot (-\ln 3) \right) + V \\ &= \left(-\frac{1}{2} \cdot \ln 3 \cdot \ln 2 + \frac{1}{2} \cdot \ln 2 \cdot \ln 3 \right) + V \\ &= V. \end{aligned}$$

- To show $T = X$, consider the substitution $x = -\frac{1}{2} \ln t$, and hence $t = e^{-2x}$.

  When $t = \frac{1}{2}$, $x = -\frac{1}{2} \ln \frac{1}{2} = \frac{1}{2} \ln 2$, and when $t = \frac{1}{3}$, $x = -\frac{1}{2} \ln \frac{1}{3} = \frac{1}{2} \ln 3$.

We have $\mathrm{d}x = -\frac{\mathrm{d}t}{2t}$, and hence

$$\begin{aligned}
T &= \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t} \,\mathrm{d}t \\
&= \int_{\frac{1}{2}\ln 3}^{\frac{1}{2}\ln 2} \frac{\operatorname{artanh} e^{-2x}}{t} \cdot (-2t) \,\mathrm{d}x \\
&= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} 2 \operatorname{artanh} e^{-2x} \,\mathrm{d}x \\
&= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln \left(\frac{1+e^{-2x}}{1-e^{-2x}} \right) \mathrm{d}x \\
&= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln \left(\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}} \right) \mathrm{d}x \\
&= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln \coth x \,\mathrm{d}x \\
&= X.
\end{aligned}$$