2011.3 Question 5

Since we have

$$\tan\theta = \frac{y}{x} \implies \theta = \arctan\frac{y}{x} + k\pi$$

for some $k \in \mathbb{Z}$, differentiating with respect to t gives us

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\mathrm{d}\frac{y}{x}}{\mathrm{d}t} = \frac{x^2}{x^2+y^2} \cdot \frac{x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}}{x^2} = \frac{x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}}{r^2}.$$

Hence,

$$\frac{1}{2}\int r^2 \,\mathrm{d}\theta = \frac{1}{2}\int r^2 \cdot \frac{x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}}{r^2} \cdot \mathrm{d}t = \frac{1}{2}\int \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t,$$

as desired.

The coordinates of A and B are

$$A\left(x - a\cos t, y - a\sin t\right), B\left(x + b\cos t, y + b\sin t\right).$$

Hence, we have

$$\begin{aligned}
[A] &= \frac{1}{2}\int_0^{2\pi} \left(x_A\frac{\mathrm{d}y_A}{\mathrm{d}t} - y_A\frac{\mathrm{d}x_A}{\mathrm{d}y}\right)\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} \left[(x - a\cos t)\left(\frac{\mathrm{d}y}{\mathrm{d}t} - a\cos t\right) - (y - a\sin t)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + a\sin t\right)\right]\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t - \frac{a}{2}\int_0^{2\pi} \left[\cos t\left(\frac{\mathrm{d}y}{\mathrm{d}t} + x\right) + \sin t\left(y - \frac{\mathrm{d}x}{\mathrm{d}t}\right)\right]\mathrm{d}t \\
&\quad + \frac{a^2}{2}\int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right)\mathrm{d}t \\
&= [P] - af + \frac{a^2}{2}\int_0^{2\pi} \mathrm{d}t \\
&= [P] - af + 2\pi \cdot \frac{a^2}{2} \\
&= [P] + \pi a^2 - af,
\end{aligned}$$

as desired.

Similarly,

$$\begin{aligned}
[B] &= \frac{1}{2}\int_0^{2\pi} \left(x_B\frac{\mathrm{d}y_B}{\mathrm{d}t} - y_B\frac{\mathrm{d}x_B}{\mathrm{d}y}\right)\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} \left[(x + b\cos t)\left(\frac{\mathrm{d}y}{\mathrm{d}t} + b\cos t\right) - (y + b\sin t)\left(\frac{\mathrm{d}x}{\mathrm{d}t} - b\sin t\right)\right]\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t + \frac{b}{2}\int_0^{2\pi} \left[\cos t\left(\frac{\mathrm{d}y}{\mathrm{d}t} + x\right) + \sin t\left(y - \frac{\mathrm{d}x}{\mathrm{d}t}\right)\right]\mathrm{d}t \\
&\quad + \frac{b^2}{2}\int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right)\mathrm{d}t \\
&= [P] + bf + \frac{b^2}{2}\int_0^{2\pi} \mathrm{d}t \\
&= [P] + bf + 2\pi \cdot \frac{b^2}{2} \\
&= [P] + \pi b^2 + bf.
\end{aligned}$$

Since over $t \in [0, 2\pi]$, A and B both trace over $\mathcal{D}$, we must have

$$[A] = [B],$$

and hence

$$\pi a^2 - af = \pi b^2 + bf,$$

which means

$$\pi(a+b)(a-b) = (a+b)f,$$

and hence

$$f = (a-b)\pi,$$

and therefore

$$[A] = [B] = [P] + ab\pi.$$

The area between the curves $\mathcal{C}$ and $\mathcal{D}$ is represented as $[A] - [P]$ or $[B] - [P]$, and hence this area is πab, as desired.