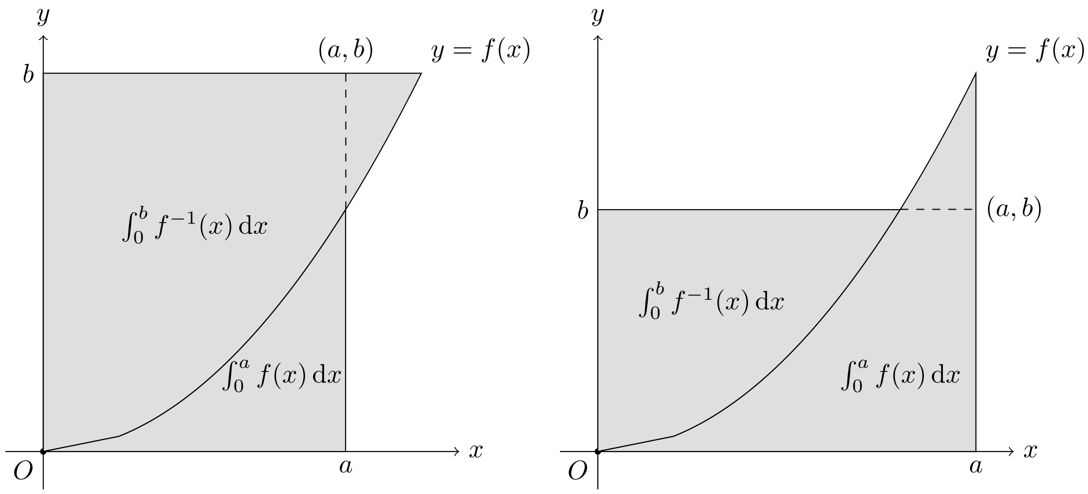
2011.3 Question 4

1. The following two diagrams shows the cases $a < b$ and $a > b$ respectively.



   In both cases, the shaded area is greater than the area of the rectangle formed by $(0,0)$, $(a,0)$, (a,b) and $(0,b)$, leading to the inequality. The equal sign holds when $b = f(a)$.

2. Since $f(x) = x^{p-1}$, we must have $x = f^{-1}(x)^{p-1}$, and hence $f^{-1}(x) = x^{\frac{1}{p-1}}$. Hence,

$$\int_0^a f(x)\,\mathrm{d}x = \frac{1}{p}\left[x^p\right]_0^a = \frac{a^p}{p}.$$

   Since $\frac{1}{p} + \frac{1}{q} = 1$, we must have $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$, and

$$q = \frac{p}{p-1},$$

   and hence

$$f^{-1}(x) = x^{q-1},$$

   which gives

$$\int_0^b f^{-1}(x)\,\mathrm{d}x = \frac{b^q}{q}.$$

   Since f is a polynomial, it must be continuous. $f(0) = 0^{p-1} = 0$, and

$$f'(x) = (p-1)x^{p-2}$$

   is always non-negative for $x \geq 0$, we must have by the original inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

   as desired.

3. Consider the function $f(x) = \sin x$. First, f is continuous, and

$$f'(x) = \cos x$$

   is always positive for $0 \leq x \leq \frac{1}{2}\pi$. We notice

$$\int_0^a f(x)\,\mathrm{d}x = \left[-\cos x\right]_0^a = 1 - \cos a,$$

and $f^{-1}(x) = \arcsin x$, and hence for $0 \leq b \leq 1$,

$$\begin{aligned}\int_0^b f^{-1}(x)\,\mathrm{d}x &= \int_0^b \arcsin(x)\,\mathrm{d}x \\ &= [x \arcsin x]_0^b - \int_0^b x \cdot \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &= \left[x \arcsin x + \sqrt{1-x^2}\right]_0^b \\ &= b \arcsin b + \sqrt{1-b^2} - 1.\end{aligned}$$

Hence, using the given inequality,

$$ab \leq b \arcsin b + \sqrt{1-b^2} - 1 + 1 - \cos a = b \arcsin b + \sqrt{1-b^2} - \cos a,$$

as desired.

Let $a = 0$ and $b = t^{-1}$. Since $t \geq 1$, we have $0 < b \leq 1$, and hence

$$0 \leq t^{-1} \arcsin t^{-1} + \sqrt{1-t^{-2}} - \cos 0.$$

Multiplying both sides by t, and noticing $\cos 0 = 1$, we have

$$0 \leq \arcsin t^{-1} + \sqrt{t^2-1} - t,$$

and hence

$$\arcsin t^{-1} \geq t - \sqrt{t^2-1},$$

as desired.