## 2011.3 Question 3

We have

$$\begin{aligned}a(x-\alpha)^3+b(x-\beta)^3&=ax^3-3a\alpha x^2+3a\alpha^2x-a\alpha^3+bx^3-3b\beta x^2+3b\beta^2x-b\beta^3\\&=(a+b)x^3-3(a\alpha+b\beta)x^2+3(a\alpha^2+b\beta^2)x-(a\alpha^3+b\beta^3).\end{aligned}$$

By comparing coefficients, we have

$$\begin{cases}a+b=1,\\-3(a\alpha+b\beta)=0\implies a\alpha+b\beta=0,\\3(a\alpha^2+b\beta^2)=-3p\implies a\alpha^2+b\beta^2=-p,\\-(a\alpha^3+b\beta^3)=q\implies a\alpha^3+b\beta^3=-q.\end{cases}$$

The first pair of equation solve to

$$(a,b)=\left(-\frac{\beta}{\alpha-\beta},\frac{\alpha}{\alpha-\beta}\right).$$

Putting this into the third equation, we can see

$$\begin{aligned}\text{LHS}&=\frac{\beta}{\beta-\alpha}\cdot\alpha^2-\frac{\alpha}{\beta-\alpha}\cdot\beta^2\\&=\frac{\alpha\beta(\alpha-\beta)}{\beta-\alpha}\\&=-\alpha\beta\\&=-\frac{p^2}{p}\\&=-p\\&=\text{RHS},\end{aligned}$$

using Vieta's Theorem for $\alpha\beta$, and for the final one,

$$\begin{aligned}\text{LHS}&=\frac{\beta}{\beta-\alpha}\cdot\alpha^3-\frac{\alpha}{\beta-\alpha}\cdot\beta^3\\&=\frac{\alpha\beta(\alpha^2-\beta^2)}{\beta-\alpha}\\&=-\frac{\alpha\beta(\alpha+\beta)(\beta-\alpha)}{\beta-\alpha}\\&=-\alpha\beta(\alpha+\beta)\\&=-\frac{p^2}{p}\cdot\left(-\frac{-q}{p}\right)\\&=-p\cdot\frac{q}{p}\\&=-q\\&=\text{RHS},\end{aligned}$$

using Vieta's Theorem for $\alpha\beta$ and $\alpha+\beta$. Hence, this means for $\alpha,\beta$ being solutions to $pt^2-qt+p^2=0$ and

$$(a,b)=\left(-\frac{\beta}{\alpha-\beta},\frac{\alpha}{\alpha-\beta}\right),$$

we have

$$x^3-3px+q=a(x-\alpha)^3+b(x-\beta)^3.$$

In this case here, we have $p=8$ and $q=48$. Hence, the quadratic equation is

$$8t^2-48t+8^2=8(t^2-6t+8)=8(t-2)(t-4)=0,$$

which solves to $(\alpha, \beta) = (2, 4)$ or $(\alpha, \beta) = (4, 2)$. Without loss of generality, let $(\alpha, \beta) = (2, 4)$, and hence

$$(a, b) = \left(-\frac{\beta}{\alpha - \beta}, \frac{\alpha}{\alpha - \beta}\right) = \left(-\frac{4}{2-4}, \frac{2}{2-4}\right) = (2, -1),$$

Hence, the original cubic equation

$$x^3 - 24x + 48 = 0$$

can be simplified to

$$2(x-2)^3 - (x-4)^3 = 0.$$

Hence,

$$2(x-2)^3 = (x-4)^3,$$

and we have

$$2^{\frac{1}{3}}(x-2) = \omega^n (x-4),$$

for $n = 0, 1, 2$ and $\omega = \exp\left(\frac{2\pi i}{3}\right)$.

Rearranging gives us

$$x = \frac{2\left(2\omega^n - 2^{\frac{1}{3}}\right)}{\omega^n - 2^{\frac{1}{3}}}$$

When $n = 0$, $\omega^n = 1$, and hence

$$x = \frac{2\left(2 - 2^{\frac{1}{3}}\right)}{1 - 2^{\frac{1}{3}}}.$$

The other two solutions

$$x = \frac{2\left(2\omega - 2^{\frac{1}{3}}\right)}{\omega - 2^{\frac{1}{3}}}, x = \frac{2\left(2\omega^2 - 2^{\frac{1}{3}}\right)}{\omega^2 - 2^{\frac{1}{3}}}.$$

This equation reduces to

$$x^3 - 3r^2x + 2r^3 = 0.$$

This can be factorised to

$$(x-r)(x^2 + rx - 2r^2) = (x-r)^2(x+2r)$$

and the solutions are

$$x_{1,2} = r, x_3 = -2r.$$