2011.3 Question 2

By definition,

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

where $a_n = 1$.

Hence,

$$\begin{aligned} q^{n-1} f\left(\frac{p}{q}\right) &= q^{n-1} \sum_{k=0}^{n} a_k \left(\frac{p}{q}\right)^k \\ &= q^{n-1} \sum_{k=0}^{n} a_k p^k q^{-k} \\ &= \sum_{k=0}^{n} a_k p^k q^{n-k-1}. \end{aligned}$$

For the terms with $k = 0, 1, 2, \dots, n-1$, we have $n - k - 1 \geq 0$ and hence the terms $a_k p^k q^{n-k-1}$ is an integer, and hence the sum from $k = 0$ to $k = n - 1$ is an integer as well.

If $\frac{p}{q}$ is a rational root of f, $f\left(\frac{p}{q}\right) = 0$, and since all the rest of the terms are integers, the term where $k = n$ must be an integer as well. When $k = n$,

$$a_k p^k q^{n-k-1} = a_n p^n q^{-1} = \frac{p^n}{q}$$

must be an integer. But since p and q are co-prime, this can be an integer if and only if $q = 1$.

Therefore, $\frac{p}{q} = p$ is an integer as well, and any rational root to $f(x) = 0$ must be an integer.

1. Consider the polynomial $f(x) = x^n - 2$. The nth root of 2 must satisfy $1 < \sqrt[n]{2} < 2$, for $n \geq 2$. This is because $1^n = 1 < 2$ and $2^n = 2 \cdot 2^{n-1} > 21 = 2$.

   The nth root of 2 is a root to f. If it is rational, then it must be integer. But $1 < \sqrt[n]{2} < 2$ and so the nth root of 2 cannot be an integer. Therefore, it must be irrational.

2. Consider the polynomial $f(x) = x^3 - x + 1$. If the roots to this polynomial are rational, then they must be integer.

   Under modulo 2, $x^3 \equiv x$ since $1^3 \equiv 1$ and $0^3 \equiv 0$. Hence, $f(x) \equiv x^3 - x + 1 \equiv 0 + 1 \equiv 1$ modulo 2. This means there is no integer root to $f(x) = 0$ since the right-hand side is congruent to 0 modulo 2, and hence there are no rational roots.

3. Consider the polynomial $f(x) = x^n - 5x + 7$. If the roots to this polynomial are rational, then they must be integer.

   For $n \geq 2$, under modulo 2, $x^n \equiv 5x$ since $1^n \equiv 1 \equiv 5 \equiv 5 \cdot 1$ and $0^n \equiv 0 \equiv 5 \cdot 0$. Hence, $f(x) \equiv x^n - 5x + 7 \equiv 0 + 7 \equiv 1$ modulo 2. This means there is no integer root to $f(x) = 0$ since the right-hand side is congruent to 0 modulo 2, and hence there are no rational roots.