2011.3 Question 13

1. We first find the expression given by the question.

$$\begin{aligned}
\frac{\mathrm{P}(X=r+1)}{\mathrm{P}(X=r)} &= \frac{\left(\frac{b}{n}\right)^{r+1}\left(\frac{n-b}{n}\right)^{k-r-1}\binom{k}{r+1}}{\left(\frac{b}{n}\right)^{r}\left(\frac{n-b}{n}\right)^{k-r}\binom{k}{r}} \\
&= \frac{b/n}{(n-b)/n}\cdot\frac{\frac{k!}{(r+1)!(k-r-1)!}}{\frac{k!}{r!(k-r)!}} \\
&= \frac{b}{n-b}\cdot\frac{r!(k-r)!}{(r+1)!(k-r-1)!} \\
&= \frac{b}{n-b}\cdot\frac{k-r}{r+1} \\
&= \frac{b}{n-b}\cdot\left(\frac{k+1}{r+1}-1\right),
\end{aligned}$$

and we can see that this decreases as r increases.

If the most probable number of black balls in the sample is unique (let it be r_0), then we have

$$\mathrm{P}(X=r_0+1)<\mathrm{P}(X=r_0) \iff \frac{\mathrm{P}(X=r_0+1)}{\mathrm{P}(X=r_0)}<1,$$

and

$$\mathrm{P}(X=r_0-1)<\mathrm{P}(X=r_0) \iff \frac{\mathrm{P}(X=r_0)}{\mathrm{P}(X=r_0-1)}>1,$$

This means r_0 is the minimal solution to the inequality

$$\frac{\mathrm{P}(X=r+1)}{\mathrm{P}(X=r)}<1.$$

This could be simplified to

$$\begin{aligned}
\frac{\mathrm{P}(X=r+1)}{\mathrm{P}(X=r)} &< 1 \\
\frac{b}{n-b}\left(\frac{k+1}{r+1}-1\right) &< 1 \\
\frac{k+1}{r+1}-1 &< \frac{n-b}{b} \\
\frac{k+1}{r+1} &< \frac{n}{b} \\
r+1 &> \frac{b(k+1)}{n} \\
r &> \frac{b(k+1)}{n}-1,
\end{aligned}$$

and hence

$$r_0=\left\lfloor\frac{b(k+1)}{n}\right\rfloor.$$

It is not unique when there exists some r where

$$\frac{\mathrm{P}(X=r_0+1)}{\mathrm{P}(X=r_0)}=1,$$

which means there exists an integer r such that

$$r=\frac{b(k+1)}{n}-1.$$

This happens if and only if $n \mid b(k+1)$.

2. Let Y be the number of black balls in the sample. Similarly, we have

$$\begin{aligned}
\frac{\mathrm{P}(Y=r+1)}{\mathrm{P}(Y=r)} &= \frac{\frac{\binom{b}{r+1}\cdot\binom{n-b}{k-r-1}}{\binom{n}{k}}}{\frac{\binom{b}{r}\cdot\binom{n-b}{k-r}}{\binom{n}{k}}} \\
&= \frac{\frac{b!}{(r+1)!(b-r-1)!}\cdot\frac{(n-b)!}{(k-r-1)!(n+r-k-b+1)!}}{\frac{b!}{r!(b-r)!}\cdot\frac{(n-b)!}{(k-r)!(n+r-k-b)!}} \\
&= \frac{r!(b-r)!(k-r)!(n+r-k-b)!}{(r+1)!(b-r-1)!(k-r-1)!(n+r-k-b+1)!} \\
&= \frac{(b-r)\cdot(k-r)}{(r+1)\cdot(n+r-k-b+1)}.
\end{aligned}$$

The most probable number of black balls is the smallest solution to

$$\begin{aligned}
\frac{(b-r)\cdot(k-r)}{(r+1)\cdot(n+r-k-b+1)} &< 1 \\
(b-r)(k-r) &< (r+1)(n+r-k-b+1) \\
bk-rk-bk+r^2 &< nr+r^2-rk-bk+r+n+r-k-b+1 \\
(n+2)r &> bk+k+b-1-n \\
r &> \frac{bk+k+b-1-n}{n+2} \\
&= \frac{(n+1)(k+1)}{n+2}-1.
\end{aligned}$$

This means the most probable number of black balls, given its uniqueness, is

$$\left\lfloor \frac{(b+1)(k+1)}{(n+2)} \right\rfloor.$$

It is not unique when

$$\frac{(n+1)(k+1)}{n+2}-1$$

is an integer, if and only if

$$(n+2) \mid (n+1)(k+1).$$