2011.3 Question 12

By differentiation, we have
$$[G(H(t))]' = G'(H(t)) \cdot H'(t).$$

Hence, we have

$$\begin{aligned}\mathrm{E}(Y) &= [G(H(t))]'|_{t=1}\\ &= G'(H(1)) \cdot H'(1)\\ &= G'(1) \cdot H'(1)\\ &= \mathrm{E}(N) \cdot \mathrm{E}(X_i).\end{aligned}$$

By differentiating twice, we have

$$[G(H(t))]'' = G''(H(t)) \cdot H'(t) \cdot H'(t) + G'(H(t)) \cdot H''(t).$$

Hence, we have

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{E}(Y(Y-1)) + \mathrm{E}(Y) - \mathrm{E}(Y)^2\\ &= [G(H(t))]''|_{t=1} + \mathrm{E}(Y) - \mathrm{E}(Y)^2\\ &= G''(H(1)) \cdot H'(1) \cdot H'(1) + G'(H(1)) \cdot H''(1) + \mathrm{E}(Y) - \mathrm{E}(Y)^2\\ &= G''(1) \cdot H'(1)^2 + G'(1) \cdot H''(1) + \mathrm{E}(Y) - \mathrm{E}(Y)^2\\ &= \mathrm{E}(N(N-1)) \cdot \mathrm{E}(X_i)^2 + \mathrm{E}(N) \cdot \mathrm{E}(X_i(X_i-1)) + \mathrm{E}(Y) - \mathrm{E}(Y)^2\\ &= \left[\mathrm{Var}(N) + \mathrm{E}(N)^2 - \mathrm{E}(N)\right] \cdot \mathrm{E}(X_i)^2 + \mathrm{E}(N) \cdot \left[\mathrm{Var}(X_i) + \mathrm{E}(X_i^2) - \mathrm{E}(X_i)\right]\\ &\quad + \mathrm{E}(N) \cdot \mathrm{E}(X_i) - \mathrm{E}(N)^2 \cdot \mathrm{E}(X_i)^2\\ &= \mathrm{Var}(N)\,\mathrm{E}(X_i)^2 + \mathrm{E}(N)\,\mathrm{Var}(X_i).\end{aligned}$$

As defined, we have $N \sim \mathrm{Geo}\left(\frac{1}{2}\right)$, and hence

$$G(t) = \frac{\frac{1}{2} \cdot t}{1 - \left(1 - \frac{1}{2}\right) t} = \frac{t}{2-t},$$

and

$$\mathrm{E}(N) = 1/\frac{1}{2} = 2, \mathrm{Var}(N) = \frac{1 - \frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 2.$$

We have $X_i \sim \mathrm{B}\left(1, \frac{1}{2}\right)$, and hence

$$H(t) = \frac{1}{2} \cdot t^0 + \frac{1}{2} \cdot t^1 = \frac{1}{2} \cdot (1+t),$$

and

$$\mathrm{E}(X_i) = 1 \cdot \frac{1}{2} = \frac{1}{2}, \mathrm{Var}(X_i) = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Hence, for $Y = \sum_{i=1}^{N} X_i$, we have

$$\text{p.g.f.}_Y(t) = G(H(t)) = \frac{\frac{1}{2}(1+t)}{2 - \frac{1}{2}(1+t)} = \frac{1+t}{3-t},$$

and by the formula for expectation and variance, we have

$$\mathrm{E}(Y) = \mathrm{E}(N)\,\mathrm{E}(X_i) = 2 \cdot \frac{1}{2} = 1,$$

and

$$\mathrm{Var}(Y) = \mathrm{Var}(N) \cdot \mathrm{E}(X_i)^2 + \mathrm{E}(N) \cdot \mathrm{Var}(X_i) = 2 \cdot \left(\frac{1}{2}\right)^2 + 2 \cdot \frac{1}{4} = 1.$$

By expressing the probability generating function of Y as a power series, we notice that

$$\begin{aligned}
\text{p.g.f.}_Y(t) &= \frac{1+t}{3-t} \\
&= -1 + \frac{4}{3-t} \\
&= -1 + \frac{4}{3} \cdot \frac{1}{1-\frac{t}{3}} \\
&= -1 + \frac{4}{3} \sum_{r=0}^{\infty} \left(\frac{t}{3}\right)^r \\
&= -1 + \frac{4}{3} + \frac{4}{3} \sum_{r=1}^{\infty} 3^{-r} \cdot t^r \\
&= \frac{1}{3} + \frac{4}{3} \sum_{r=1}^{\infty} 3^{-r} \cdot t^r,
\end{aligned}$$

and hence

$$\mathrm{P}(Y = y) = \begin{cases} \frac{1}{3}, & y = 0, \\ \frac{4}{3^{y+1}}, & \text{otherwise.} \end{cases}$$