2011.3 Question 1

1. By rearrangement, we have
$$\frac{\mathrm{d}u}{u} = \left(1 + \frac{1}{x+1}\right)\mathrm{d}x,$$
and hence by integration,
$$\ln|u| = x + \ln|x+1| + C.$$
This gives
$$u = C(x+1)e^x$$
as the general solution.

2. Since $y = ze^{-x}$, we must have
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x}e^{-x} - ze^{-x} = \left(\frac{\mathrm{d}z}{\mathrm{d}x} - z\right)e^{-x},$$
and
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{-x} - 2\frac{\mathrm{d}z}{\mathrm{d}x}e^{-x} + ze^{-x} = \left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + z\right)e^{-x}.$$
Hence, the original differential equation can be simplified:
$$\begin{aligned} (x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y &= 0 \\ (x+1)\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + z\right)e^{-x} + x\left(\frac{\mathrm{d}z}{\mathrm{d}x} - z\right)e^{-x} - ze^{-x} &= 0 \\ (x+1)\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + z\right) + x\left(\frac{\mathrm{d}z}{\mathrm{d}x} - z\right) - z &= 0 \\ (x+1)\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - (x+2)\frac{\mathrm{d}z}{\mathrm{d}x} &= 0, \end{aligned}$$
which is a first-order differential equation for $\frac{\mathrm{d}z}{\mathrm{d}x}$.
Hence, from part (i), we have the general solution to this differential equation is
$$\frac{\mathrm{d}z}{\mathrm{d}x} = C(x+1)e^x,$$
and hence by integration
$$z = C\int(x+1)e^x\,\mathrm{d}x = C\left[\int x\,\mathrm{d}e^x + \int e^x\,\mathrm{d}x\right] = C[xe^x - e^x + e^x] + D.$$
Therefore, $y = ze^{-x} = De^{-x} + Cx$. Let $A = C$ and $B = D$ and this is exactly what is desired.

3. The complementary function is the differential equation solved in the previous part. For the complementary function, consider $y = ax^2 + b$, and hence $\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2a$. Hence,
$$2a(x+1) + x\cdot 2ax - ax^2 - c = ax^2 + 2ax + (2a - c) = x^2 + 2x + 1.$$
Hence, $a = 1$ and $c = 1$ giving $y = x^2 + 1$ is a particular integral.
Therefore, the general solution to the differential equation is
$$y = Ax + Be^{-x} + x^2 + 1.$$