2010.3 Question 8

Since $P(x) = Q(x)R'(x) - Q'(x)R(x)$, we notice that

$$\frac{P(x)}{Q(x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{R(x)}{Q(x)}.$$

Hence,

$$\int \frac{P(x)}{Q(x)^2}\,\mathrm{d}x = \frac{R(x)}{Q(x)} + C$$

where C is a real constant.

1. Since $R(x) = a + bx + cx^2$, we have $R'(x) = b + 2cx$. We let $P(x) = 5x^2 - 4x - 3$ and $Q(x) = 1 + 2x + 3x^2$, and hence $Q'(x) = 6x + 2$.

   Hence,
   $$5x^2 - 4x - 3 = (1 + 2x + 3x^2)(b + 2cx) - (6x + 2)(a + bx + cx^2).$$

   Notice that
   $$\begin{aligned}\text{RHS} &= [6cx^3 + (3b + 4c)x^2 + (2b + 2c)x + b] - [6cx^3 + (6b + 2c)x^2 + (6a + 2b)x + 2a] \\ &= (-3b + 2c)x^2 + (-6a + 2c) + (-2a + b).\end{aligned}$$

   Hence, we have
   $$\begin{cases} & -3b + 2c = 5, \\ -6a + 2c = -4 \iff & 3a - c = 2, \\ & -2a + b = -3. \end{cases}$$

   Notice that
   $$1 \cdot (-3b + 2c) + 2 \cdot (3a - c) + 3 \cdot (-2a + b) = 0,$$
   and
   $$1 \cdot 5 + 2 \cdot 2 - 3 \cdot 3 = 0,$$
   which means that these three equations are linearly dependent. Hence, let $a = 0$, and hence $b = -3$, $c = -2$, $R(x) = -3x - 2x^2$, which gives
   $$\int \frac{5x^2 - 4x - 3}{(1 + 2x + 3x^2)^2}\,\mathrm{d}x = \frac{-3x - 2x^2}{1 + 2x + 3x^2} + C_1.$$

   Letting $a = 1$, and hence $b = -1$, $c = 1$, $R(x) = 1 - x + x^2$, which gives
   $$\int \frac{5x^2 - 4x - 3}{(1 + 2x + 3x^2)^2}\,\mathrm{d}x = \frac{1 - x + x^2}{1 + 2x + 3x^2} + C_2.$$

   Notice that
   $$\frac{1 - x + x^2}{1 + 2x + 3x^2} - \frac{-3x - 2x^2}{1 + 2x + 3x^2} = \frac{1 + 2x + 3x^2}{1 + 2x + 3x^2} = 1,$$
   and the integrals just differ by a constant. Different choices of (a, b, c) lead to results which only differ by a constant.

2. The differential equation we are attempting to solve is equivalent to
   $$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\sin x - 2\cos x}{1 + \cos x + 2\sin x}y = \frac{5 - 3\cos x + 4\sin x}{1 + \cos x + 2\sin x}.$$

   The integrating factor is
   $$\begin{aligned}I(x) &= \exp\int \frac{\sin x - 2\cos x}{1 + \cos x + 2\sin x}\,\mathrm{d}x \\ &= \exp\int -\frac{\mathrm{d}(1 + \cos x + 2\sin x)}{1 + \cos x + 2\sin x} \\ &= \exp(-\ln|1 + \cos x + 2\sin x|) \\ &= \frac{1}{1 + \cos x + 2\sin x},\end{aligned}$$

and hence
$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{y}{1+\cos x+2\sin x}=\frac{5-3\cos x+4\sin x}{(1+\cos x+2\sin x)^2}.$$

Let $Q(x)=1+\cos x+2\sin x$, and let $P(x)=5-3\cos x+4\sin x$. We have $Q'(x)=2\cos x-\sin x$ Set $R(x)=a+b\sin x+c\cos x$ for some real constant a,b and c. We have $R'(x)=b\cos x-c\sin x$. Hence,

$$5-3\cos x+4\sin x=(1+\cos x+2\sin x)(b\cos x-c\sin x)-(2\cos x-\sin x)(a+b\sin x+c\cos x).$$

We expand the brackets on the right-hand side, and we have

$$\begin{aligned}\text{RHS}&=b\cos x-c\sin x+b\cos^2 x-c\cos x\sin x+2b\sin x\cos x-2c\sin^2 x\\&\quad-2a\cos x-2b\sin x\cos x-2c\cos^2 x+a\sin x+b\sin^2 x+c\sin x\cos x\\&=(a-c)\sin x+(b-2a)\cos x+(b-2c)\left(\sin^2 x+\cos^2 x\right)\\&=(b-2c)+(a-c)\sin x+(b-2a)\cos x,\end{aligned}$$

and hence by comparing coefficients, we have

$$\begin{cases}b-2c=5,\\a-c=4,\\-2a+b=-3.\end{cases}$$

Notice that
$$1\cdot(b-2c)+(-2)\cdot(a-c)+(-1)\cdot(-2a+b)=0,$$
and
$$1\cdot 5+(-2)\cdot 4+(-1)\cdot(-3)=0,$$
so the system of linear equations is linearly dependent. Hence, set $a=0$, and we have $b=-3,c=-4$, and we have $R(x)=-3\sin x-4\cos x$.

Hence,

$$\begin{aligned}\int\frac{5-3\cos x+4\sin x}{(1+\cos x+2\sin x)^2}&=\int\frac{P(x)}{Q(x)^2}\,\mathrm{d}x\\&=\frac{R(x)}{Q(x)}+C\\&=-\frac{3\sin x+4\cos x}{1+\cos x+2\sin x}+C,\end{aligned}$$

and hence
$$\frac{y}{1+\cos x+2\sin x}=-\frac{3\sin x+4\cos x}{1+\cos x+2\sin x}+C,$$
which means the general solution to the differential equation is

$$y=-(3\sin x+4\cos x)+C(1+\cos x+2\sin x).$$