2010.3 Question 7

Since $y = \cos(m \arcsin x)$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{m\sin(m\arcsin x)}{\sqrt{1-x^2}},$$

and

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\frac{m^2\cos(m\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}\cdot\sqrt{1-x^2} - m\sin(m\arcsin x)\cdot(-x)\cdot\frac{1}{\sqrt{1-x^2}}}{1-x^2}\\ &= -\frac{m}{1-x^2}\left(m\cos(m\arcsin x) + x\sin(m\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}\right).\end{aligned}$$

Hence, the left-hand side of the differential equation reduces to

$$\begin{aligned}&(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2y\\ &= -m\cdot\left(m\cos(m\arcsin x) + x\sin(m\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}\right)\\ &\quad+\frac{mx\sin(m\arcsin x)}{\sqrt{1-x^2}} + m^2\cos(m\arcsin x)\\ &= -m^2\cos(m\arcsin x) + m^2\cos(m\arcsin x)\\ &\quad-\frac{mx\sin(m\arcsin x)}{\sqrt{1-x^2}} + \frac{mx\sin(m\arcsin x)}{\sqrt{1-x^2}}\\ &= 0,\end{aligned}$$

as desired.

Differentiating both sides of this equation with respect to x, we get

$$(-2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1-x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - \frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + m^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

which reduces to

$$(1-x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (m^2-1)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Differentiating both sides with respect to x again, we get

$$(-2x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (1-x^2)\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - 3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (m^2-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0,$$

which reduces to

$$(1-x^2)\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - 5x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (m^2-4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.$$

The conjecture is for all $n \geq 0$,

$$(1-x^2)\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} - (2n+1)\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} + (m^2-n^2)\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = 0.$$

The base case where $n = 0$ is already shown. We show the inductive step. Assume this statement is true for some $n = k$, i.e.

$$(1-x^2)\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} - (2k+1)x\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} + (m^2-k^2)\frac{\mathrm{d}^ky}{\mathrm{d}x^k} = 0.$$

Differentiating both sides with respect to x gives

$$(-2x)\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} + (1-x^2)\frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - (2k+1)\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} - (2k+1)x\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} + (m^2-k^2)\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = 0,$$

which reduces to

$$(1-x^2)\frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - (2k+3)x\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} + (m^2-(k+1)^2)\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = 0.$$

This is precisely the statement for when $n = k + 1$.

Hence, by the principle of mathematical induction, the conjecture holds for all integers $n \geq 0$.

Now, we evaluate this at $x = 0$, and we have

$$\left.\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}\right|_{x=0} + (m^2 - n^2)\left.\frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right|_{x=0} = 0$$

for all $n \geq 0$, which rearranged gives

$$\left.\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}\right|_{x=0} = (n^2 - m^2)\left.\frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right|_{x=0}.$$

Notice that

$$\left.y\right|_{x=0} = \cos(m \arcsin 0) = 1,$$

and

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = -\frac{m\sin(m\arcsin 0)}{\sqrt{1-0^2}} = 0$$

Hence,

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = (0^2 - m^2)\left.y\right|_{x=0} = -m^2,$$

and

$$\left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=0} = (1^2 - m^2)\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 0,$$

and

$$\left.\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right|_{x=0} = (2^2 - m^2)\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = -m^2(2^2 - m^2),$$

In general, we have

$$\left.\frac{\mathrm{d}^{2n+1}y}{\mathrm{d}x^{2n+1}}\right|_{x=0} = 0,$$

and

$$\left.\frac{\mathrm{d}^{2n}y}{\mathrm{d}x^{2n}}\right|_{x=0} = \prod_{k=0}^{n-1}(4k^2 - m^2) = (-1)^n \prod_{k=0}^{n-1}(m^2 - 4k^2)$$

for all integers $n \geq 0$.

Hence, the Maclaurin series for y satisfy that

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} \frac{\left.\frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right|_{x=0} x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{k=0}^{n-1}(m^2 - 4k^2)x^{2n}}{(2n)!} \\
&= 1 - \frac{m^2x^2}{2!} + \frac{m^2(m^2-2^2)x^4}{4!} - \cdots.
\end{aligned}$$

In the case where m is even, notice that when $m = 2k$, $m^2 - 4k^2 = 0$, and so for all $n \geq \frac{m}{2} + 1$,

$$\prod_{k=0}^{n-1}(m^2 - 4k^2)x^{2n} = 0,$$

and hence this infinite sum becomes finite:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{k=0}^{n-1}(m^2 - k^2)x^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\frac{m}{2}} \frac{(-1)^n \prod_{k=0}^{n-1}(m^2 - k^2)x^{2n}}{(2n)!}.
\end{aligned}$$

Now, let $x = \sin\theta$, we have $\theta = \arcsin x$ since $|\theta| < \frac{1}{2}\pi$, and $y = \cos m\theta$. Hence,

$$\cos m\theta = 1 - \frac{m^2\sin^2\theta}{2!} + \frac{m^2(m^2-2^2)\sin^4\theta}{4!} - \cdots,$$

where the sum is finite (and hence a polynomial), and the degree of this polynomial is m;