2010.3 Question 6

1. The coordinates of P_1 are
$$P_1(\cos\varphi, \sin\varphi, 0),$$
and the coordinates of Q_1 are
$$Q_1(-\sin\varphi, \cos\varphi, 0).$$
Since the rotation is about z-axis, the position of R remains unchanged
$$R_1(0, 0, 1).$$

2. This rotation axis is precisely OQ_1, since it is contained in the x-y plane, and is perpendicular to OP_1. Hence, the position of Q remains unchanged, and hence
$$Q_2(-\sin\varphi, \cos\varphi, 0).$$
If we drop a perpendicular from P_2 to the line OP_1, and call the intersection be P'. We can see from trigonometry that
$$P_2P' = \sin\lambda,$$
and
$$OP' = \cos\lambda.$$
Hence, the x-coordinate of P_2 is $\cos\lambda\cos\varphi$, and the y-coordinate of P_2 is $\cos\lambda\sin\varphi$. The z-coordinate of P_2 is $\sin\lambda$, and hence
$$P_2(\cos\varphi\cos\lambda, \sin\varphi\cos\lambda, \sin\lambda).$$
The relative positions of P, Q and R remains unchanged under rotation, and hence
$$\begin{aligned}
\mathbf{r}_{R_2} &= \mathbf{r}_{P_2} \times \mathbf{r}_{Q_2} \\
&= \begin{pmatrix} \cos\varphi\cos\lambda \\ \sin\varphi\cos\lambda \\ \sin\lambda \end{pmatrix} \times \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\varphi\cos\lambda & \sin\varphi\cos\lambda & \sin\lambda \\ -\sin\varphi & \cos\varphi & 0 \end{vmatrix} \\
&= \begin{pmatrix} \sin\varphi\cos\lambda\cdot 0 - \sin\lambda\cos\varphi \\ -(\cos\varphi\cos\lambda\cdot 0 + \sin\lambda\cdot\sin\varphi) \\ \cos\varphi\cos\lambda\cdot\cos\varphi + \sin\varphi\cos\lambda\cdot\sin\varphi \end{pmatrix} \\
&= \begin{pmatrix} -\sin\lambda\cos\varphi \\ -\sin\lambda\sin\varphi \\ \cos^2\varphi\cos\lambda + \sin^2\varphi\cos\lambda \end{pmatrix} \\
&= \begin{pmatrix} -\sin\lambda\cos\varphi \\ -\sin\lambda\sin\varphi \\ \cos\lambda \end{pmatrix},
\end{aligned}$$
and hence
$$R_2(-\sin\lambda\cos\varphi, -\sin\lambda\sin\varphi, \cos\lambda).$$

3. The angle of rotation is the angle between OP_0 and OP_2, and hence
$$\begin{aligned}
\cos\theta &= \frac{\overrightarrow{OP_0}\cdot\overrightarrow{OP_2}}{\left|\overrightarrow{OP_0}\right|\cdot\left|\overrightarrow{OP_2}\right|} \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi\cos\lambda \\ \sin\varphi\cos\lambda \\ \sin\lambda \end{pmatrix} \\
&= \cos\varphi\cos\lambda,
\end{aligned}$$

as desired.

The axis of this rotation must be perpendicular to both OP_1 and OP_2, and hence their cross product

$$\begin{aligned}\overrightarrow{OP_0} \times \overrightarrow{OP_2} &= \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}\cos\varphi\cos\lambda\\ \sin\varphi\cos\lambda\\ \sin\lambda\end{pmatrix}\\ &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 0 & 0\\ \cos\varphi\cos\lambda & \sin\varphi\cos\lambda & \sin\lambda\end{vmatrix}\\ &= \begin{pmatrix}0\\ -\sin\lambda\\ \sin\varphi\cos\lambda\end{pmatrix}\end{aligned}$$

is a vector in the direction of the axis.