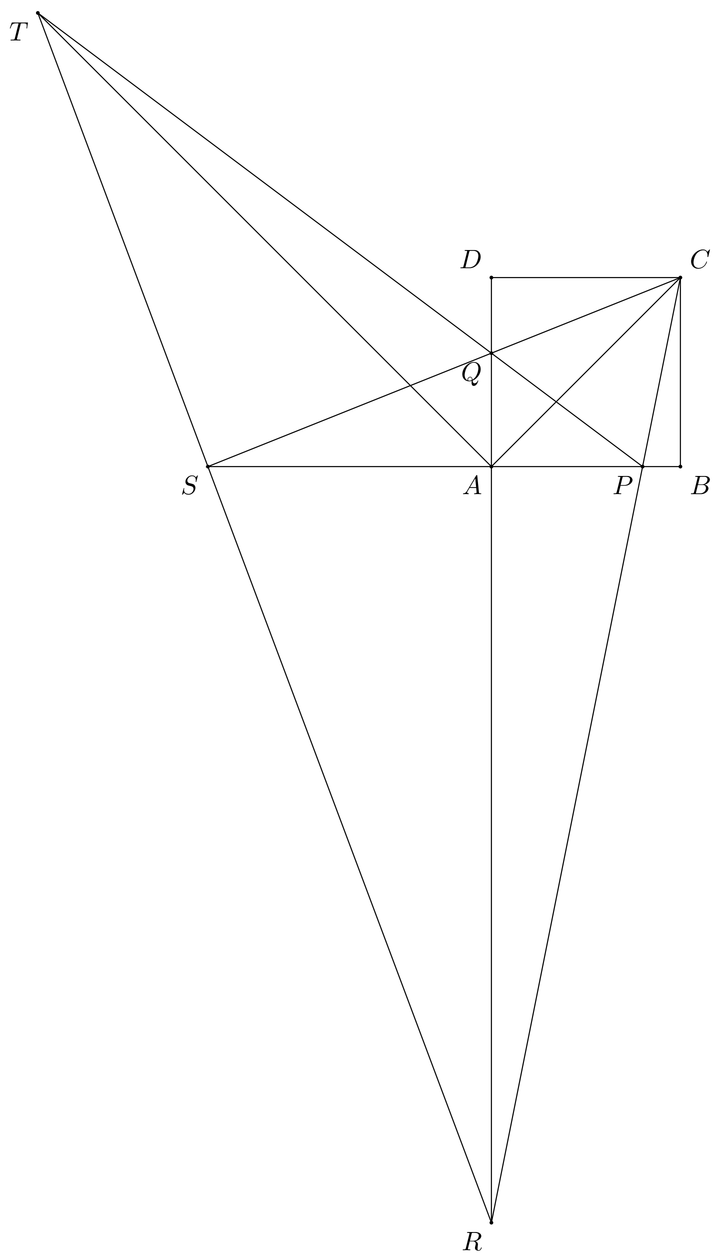
2010.3 Question 5



The line CP has equation

$$l_{CP} : y = \frac{1}{1-n}x - \frac{an}{1-n},$$

and the line DA has equation $l_{DA} : x = 0$. Hence, R has coordinates

$$R\left(0, -\frac{an}{1-n}\right).$$

The line CQ has equation

$$l_{CQ} : y = (1-m)x + am,$$

and the line BA has equation $l_{BA} : y = 0$. Hence, S has coordinates

$$S\left(-\frac{am}{1-m}, 0\right).$$

The line PQ has equation

$$l_{PQ} : y = -\frac{m}{n}x + am,$$

and the line RS has equation

$$l_{RS}: y = -\frac{n(1-m)}{m(1-n)} \cdot x - \frac{an}{1-n}.$$

Therefore, T must have x-coordinates satisfying

$$-\frac{m}{n}x + am = -\frac{n(1-m)}{m(1-n)} \cdot x - \frac{an}{1-n},$$

and hence

$$\left(\frac{n(1-m)}{m(1-n)} - \frac{m}{n}\right) x = -a\left(m + \frac{n}{1-n}\right),$$

and hence

$$\frac{n^2(1-m) - m^2(1-n)}{mn(1-n)} \cdot x = -a\left(\frac{m(1-n)+n}{1-n}\right),$$

which gives

$$\frac{(n+m-mn)(n-m)}{mn(1-n)} \cdot x = -a \cdot \frac{m+n-mn}{1-n}.$$

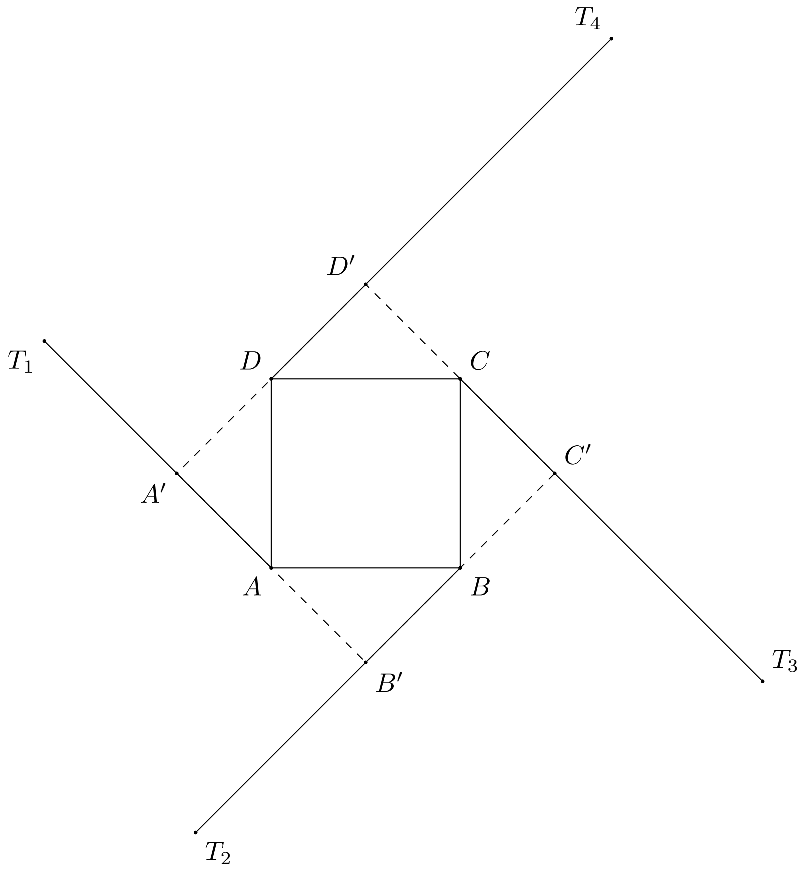
This means

$$x = \frac{amn}{m-n},$$

and hence

$$y = -\frac{m}{n} \cdot \frac{amn}{m-n} + am = \frac{-am^2 + am(m-n)}{m-n} = \frac{-amn}{m-n}.$$

This shows that line AT is the line $y = -x$, while line AC is the line $y = x$. Therefore, means that AT is perpendicular to AC.



Label the square $ABCD$ (in a counter-clockwise sequence), and find two arbitrary points P and Q on AB and AD respectively, with different distances away from A. Construct the line CP and CQ, and let their intersections with AD and AB be R and S respectively. Construct the line RS and PQ, and let them meet at T_1. We have T_1A is perpendicular to AC.

Repeating this process (rotating the labelling of A, B, C and D counter-clockwise), we will get T_2B, T_3C and T_4D, as shown in the diagrams. The square formed by these four lines is $A'B'C'D'$ (found by intersecting the lines). The new square has side length $\sqrt{a}$ equal to the length of the diameter, and hence have area $2a^2$.