2010.3 Question 2

1. We have by definition

$$\cosh a = \frac{e^a + e^{-a}}{2}.$$

Notice that

$$\begin{aligned}\frac{1}{x^2 + 2x\cosh a + 1} &= \frac{1}{x^2 + (e^a + e^{-a})x + (e^a \cdot e^{-a})} \\ &= \frac{1}{(x + e^a)(x + e^{-a})} \\ &= \left(\frac{1}{x + e^{-a}} - \frac{1}{x + e^a}\right) \cdot \frac{1}{e^a - e^{-a}},\end{aligned}$$

and hence

$$\int \frac{\mathrm{d}x}{x^2 + 2x\cosh a + 1} = \frac{\ln|x + e^{-a}| - \ln|x + e^a|}{e^a - e^{-a}} = \frac{1}{e^a - e^{-a}} \ln\left|\frac{x + e^{-a}}{x + e^a}\right|.$$

Therefore,

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{x^2 + 2x\cosh a + 1} &= \frac{1}{e^a - e^{-a}} \left[\ln\left|\frac{1 + e^{-a}}{1 + e^a}\right| - \ln\left|\frac{e^{-a}}{e^a}\right|\right] \\ &= \frac{1}{e^a - e^{-a}} \left[\ln\left|\frac{1 + e^{-a}}{e^a(1 + e^{-a})}\right| + 2a\right] \\ &= \frac{1}{e^a - e^{-a}} [-a + 2a] \\ &= \frac{a}{e^a - e^{-a}} \\ &= \frac{a}{2\sinh a}.\end{aligned}$$

2. For the first integral, we have by definition

$$\sinh a = \frac{e^a - e^{-a}}{2}.$$

Notice that

$$\begin{aligned}\frac{1}{x^2 + 2x\sinh a - 1} &= \frac{1}{x^2 + (e^a - e^{-a})x - (e^a \cdot e^{-a})} \\ &= \frac{1}{(x + e^a)(x - e^{-a})} \\ &= \left(\frac{1}{x - e^{-a}} - \frac{1}{x + e^a}\right) \cdot \frac{1}{e^a + e^{-a}},\end{aligned}$$

and hence

$$\int \frac{\mathrm{d}x}{x^2 + 2x\sinh a - 1} = \frac{\ln|x - e^{-a}| - \ln|x + e^a|}{e^a + e^{-a}} = \frac{1}{e^a + e^{-a}} \ln\left|\frac{x - e^{-a}}{x + e^a}\right|.$$

Therefore,

$$\begin{aligned}
\int_1^\infty \frac{\mathrm{d}x}{x^2 + 2x\sinh a - 1} &= \frac{1}{e^a + e^{-a}} \cdot \left[\ln\left|\frac{x - e^{-a}}{x + e^a}\right|\right]_1^\infty \\
&= \frac{1}{2\cosh a} \cdot \left[\ln 1 - \ln\left|\frac{1 - e^{-a}}{1 + e^a}\right|\right] \\
&= \frac{1}{2\cosh a} \cdot \ln \frac{1 + e^a}{1 - e^{-a}} \\
&= \frac{1}{2\cosh a} \cdot \left(a + \ln \frac{1 + e^{-a}}{1 - e^{-a}}\right) \\
&= \frac{1}{2\cosh a} \cdot \left(a + \ln \frac{e^{\frac{a}{2}} + e^{-\frac{a}{2}}}{e^{\frac{a}{2}} - e^{-\frac{a}{2}}}\right) \\
&= \frac{1}{2\cosh a} \cdot \left(a + \ln \coth \frac{a}{2}\right).
\end{aligned}$$

For the second integral, notice that

$$\frac{1}{x^4 + 2x^2\cosh a + 1} = \frac{1}{e^a - e^{-a}} \left(\frac{1}{x^2 + e^{-a}} - \frac{1}{x^2 + e^a}\right),$$

and hence

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}x}{x^4 + 2x^2\cosh a + 1} &= \frac{1}{2\sinh a} \int_0^\infty \left(\frac{1}{x^2 + e^{-a}} - \frac{1}{x^2 + e^a}\right) \mathrm{d}x \\
&= \frac{1}{2\sinh a} \left[e^{\frac{a}{2}} \arctan\left(e^{\frac{a}{2}} x\right) - e^{-\frac{a}{2}} \arctan\left(e^{-\frac{a}{2}} x\right)\right]_0^\infty \\
&= \frac{1}{2\sinh a} \left[\left(e^{\frac{a}{2}} \frac{\pi}{2} - e^{-\frac{a}{2}} \frac{\pi}{2}\right) - \left(e^{\frac{a}{2}} 0 - e^{-\frac{a}{2}} 0\right)\right] \\
&= \frac{1}{2\sinh a} \cdot (e^{\frac{a}{2}} - e^{-\frac{a}{2}}) \cdot \frac{\pi}{2} \\
&= \frac{\pi \sinh \frac{a}{2}}{2\sinh a} \\
&= \frac{\pi \sinh \frac{a}{2}}{4\sinh \frac{a}{2} \cosh \frac{a}{2}} \\
&= \frac{\pi}{4\cosh \frac{a}{2}}.
\end{aligned}$$