2010.3 Question 13

Since Z_1 and Z_2 are independent, we have

$$\operatorname{Cov}(Z_1, Z_2) = 0,$$

and hence

$$\operatorname{Corr}(Z_1, Z_2) = \frac{\operatorname{Cov}(Z_1, Z_2)}{\sqrt{\operatorname{Var}(Z_1)\operatorname{Var}(Z_2)}} = \frac{0}{\sqrt{1 \cdot 1}} = 0.$$

For Y_2, we have

$$\begin{aligned}
\mathrm{E}(Y_2) &= \mathrm{E}\left(\rho_{12} Z_1 + \sqrt{1-\rho_{12}^2} Z_2\right) \\
&= \rho_{12}\,\mathrm{E}(Z_1) + \sqrt{1-\rho_{12}^2}\,\mathrm{E}(Z_2) \\
&= \rho_{12} \cdot 0 + \sqrt{1-\rho_{12}^2} \cdot 0 \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\operatorname{Var}(Y_2) &= \operatorname{Var}(\rho_{12} Z_1 + \sqrt{1-\rho_{12}^2} Z_2) \\
&= \rho_{12}^2 \operatorname{Var}(Z_1) + (1-\rho_{12}^2)\operatorname{Var}(Z_2) \\
&= \rho_{12}^2 \cdot 1 + (1-\rho_{12}^2) \cdot 1 \\
&= 1,
\end{aligned}$$

and hence

$$\begin{aligned}
\operatorname{Corr}(Y_1, Y_2) &= \frac{\operatorname{Cov}(Y_1, Y_2)}{\sqrt{\operatorname{Var}(Y_1)\operatorname{Var}(Y_2)}} \\
&= \frac{\mathrm{E}(Y_1 Y_2) - \mathrm{E}(Y_1)\,\mathrm{E}(Y_2)}{\sqrt{1 \cdot 1}} \\
&= \mathrm{E}\left(\rho_{12} Z_1^2 + \rho_{12}\sqrt{1-\rho_{12}^2} Z_1 Z_2\right) - 0 \cdot 0 \\
&= \rho_{12}\,\mathrm{E}(Z_1^2) + \rho_{12}\sqrt{1-\rho_{12}^2}\,\mathrm{E}(Z_1 Z_2) \\
&= \rho_{12}\left(\operatorname{Var}(Z_1) + \mathrm{E}(Z_1)^2\right) + \rho_{12}\sqrt{1-\rho_{12}^2}\,\mathrm{E}(Z_1)\,\mathrm{E}(Z_2) \\
&= \rho_{12}\left(1 + 0^2\right) + \rho_{12}\sqrt{1-\rho_{12}^2} \cdot 0 \cdot 0 \\
&= \rho_{12}.
\end{aligned}$$

For Y_3, we have

$$\begin{aligned}
\operatorname{Var}(Y_3) &= \operatorname{Var}(aZ_1 + bZ_2 + cZ_3) \\
&= a^2 \operatorname{Var}(Z_1) + b^2 \operatorname{Var}(Z_2) + c^2 \operatorname{Var}(Z_3) \\
&= a^2 + b^2 + c^2 \\
&= 1,
\end{aligned}$$

and hence $a^2 + b^2 + c^2 = 1$.

For the correlation, we have

$$\begin{aligned}
\operatorname{Corr}(Y_1, Y_3) &= \frac{\operatorname{Cov}(Y_1 Y_3)}{\sqrt{\operatorname{Var}(Y_1)\operatorname{Var}(Y_3)}} \\
&= \frac{\operatorname{E}(Y_1 Y_3) - \operatorname{E}(Y_1)\operatorname{E}(Y_3)}{\sqrt{1 \cdot 1}} \\
&= \frac{\operatorname{E}(aZ_1^2 + bZ_1 Z_2 + cZ_1 Z_3) - 0 \cdot 0}{1} \\
&= a\operatorname{E}(Z_1^2) + b\operatorname{E}(Z_1 Z_2) + c\operatorname{E}(Z_1 Z_3) \\
&= a(\operatorname{Var}(Z_1) + \operatorname{E}(Z_1)^2) + b\operatorname{E}(Z_1)\operatorname{E}(Z_2) + c\operatorname{E}(Z_1)\operatorname{E}(Z_3) \\
&= a(1 + 0^2) + b \cdot 0 \cdot 0 + c \cdot 0 \cdot 0 \\
&= a \\
&= \rho_{13},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Corr}(Y_2, Y_3) &= \frac{\operatorname{Cov}(Y_2 Y_3)}{\sqrt{\operatorname{Var}(Y_2)\operatorname{Var}(Y_3)}} \\
&= \frac{\operatorname{E}(Y_2 Y_3) - \operatorname{E}(Y_2)\operatorname{E}(Y_3)}{\sqrt{1 \cdot 1}} \\
&= \frac{\operatorname{E}\left((aZ_1 + bZ_2 + cZ_3) \cdot \left(\rho_{12} Z_1 + \sqrt{1 - \rho_{12}^2 Z_2}\right)\right) - 0 \cdot 0}{1} \\
&= \operatorname{E}\left(a\rho_{12} Z_1^2 + b\sqrt{1 - \rho_{12}^2} Z_2^2\right) \\
&= a\rho_{12}(\operatorname{Var}(Z_1) + \operatorname{E}(Z_1)^2) + b\sqrt{1 - \rho_{12}^2}(\operatorname{Var}(Z_2) + \operatorname{E}(Z_2)^2) \\
&= a\rho_{12} + b\sqrt{1 - \rho_{12}^2} \\
&= \rho_{23},
\end{aligned}$$

since all the cross-term expectation is 0, i.e. for $i \neq j$, $\operatorname{E}(Z_i Z_j) = \operatorname{E}(Z_i)\operatorname{E}(Z_j) = 0$. Hence,

$$b = \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - \rho_{12}^2}},$$

and therefore,

$$c = \sqrt{1 - a^2 - b^2} = \sqrt{1 - \rho_{13}^2 - \frac{(\rho_{23} - \rho_{12}\rho_{13})^2}{1 - \rho_{12}^2}}.$$

We could have $X_i = \mu_i + \sigma_i Y_i$ for $i = 1, 2, 3$, since

$$\operatorname{E}(X_i) = \mu_i + \sigma_i \operatorname{E}(Y_i) = \mu_i + \sigma_i \cdot 0 = \mu_i,$$

and

$$\operatorname{Var}(X_i) = \sigma_i^2 \operatorname{Var}(Y_i) = \sigma_i^2 \cdot 1 = \sigma_i^2.$$

As for correlation, we notice that for any random variables U, V, we have

$$\begin{aligned}
\operatorname{Corr}(aU+b, cU+d) &= \frac{\operatorname{Cov}(aU+b, cV+d)}{\sqrt{\operatorname{Var}(aU+b)\operatorname{Var}(cV+d)}} \\
&= \frac{\operatorname{E}((aU+b)(cV+d)) - \operatorname{E}(aU+b)\operatorname{E}(cV+d)}{\sqrt{a^2\operatorname{Var}(U)c^2\operatorname{Var}(V)}} \\
&= \frac{ac\operatorname{E}(UV) + bc\operatorname{E}(V) + ad\operatorname{E}(U) + bd - (a\operatorname{E}(U)+b)(c\operatorname{E}(V)+d)}{ac\sqrt{\operatorname{Var}(U)\operatorname{Var}(V)}} \\
&= \frac{ac\operatorname{E}(UV) + bc\operatorname{E}(V) + ad\operatorname{E}(U) + bd - ac\operatorname{E}(U)\operatorname{E}(V) - bc\operatorname{E}(V) - ad\operatorname{E}(U) - bd}{ac\sqrt{\operatorname{Var}(U)\operatorname{Var}(V)}} \\
&= \frac{ac(\operatorname{E}(UV) - \operatorname{E}(U)\operatorname{E}(V))}{ac\sqrt{\operatorname{Var}(U)\operatorname{Var}(V)}} \\
&= \frac{\operatorname{Cov}(U,V)}{\sqrt{\operatorname{Var}(U)\operatorname{Var}(V)}} \\
&= \operatorname{Corr}(U,V),
\end{aligned}$$

which shows linear coding does not affect the correlation. This implies

$$\operatorname{Corr}(X_i, X_j) = \operatorname{Corr}(Y_i, Y_j) = \rho_{ij}$$

for $i \neq j$. Therefore, $X_i = \mu_i + \sigma_i Y_i$ for $i = 1, 2, 3$ satisfies the desired.