2010.3 Question 12

Since

$$S = \sum_{n=0}^{\infty} (1 + nd) r^n,$$

we have

$$\begin{aligned}
(1-r)S &= S - rS \\
&= \sum_{n=0}^{\infty} (1+nd)r^n - \sum_{n=0}^{\infty} (1+nd)r^{n+1} \\
&= \sum_{n=0}^{\infty} (1+nd)r^n - \sum_{n=1}^{\infty} (1+(n-1)d)r^n \\
&= 1 + \sum_{n=1}^{\infty} dr^n \\
&= 1 + \frac{dr}{1-r},
\end{aligned}$$

and hence

$$S = \frac{1}{1-r} + \frac{dr}{(1-r)^2},$$

as desired.

Let X be the number shots taken for Arthur to hit the target for the first time, and we have $X \sim \text{Geo}(a)$, we would like to show $\text{E}(X) = \frac{1}{a}$.

The probability mass function for X satisfies

$$\text{P}(X = x) = (1-a)^{x-1} \cdot a,$$

and hence

$$\begin{aligned}
\text{E}(X) &= \sum_{x=1}^{\infty} x \, \text{P}(X = x) \\
&= a \cdot \sum_{x=1}^{\infty} x(1-a)^{x-1} \\
&= a \cdot \sum_{x=0}^{\infty} (1+x)(1-a)^x \\
&= a \cdot \left[\frac{1}{1-(1-a)} + \frac{1 \cdot (1-a)}{(1-(1-a))^2} \right] \\
&= a \cdot \left[\frac{1}{a} + \frac{1-a}{a^2} \right] \\
&= a \cdot \frac{1}{a^2} \\
&= \frac{1}{a},
\end{aligned}$$

as desired.

Since there is a probability a of Arthur winning on a particular shot, and if Arthur did not hit (with probability $(1-a)$), then there is a probability b of Boadicea winning on the shot, and $(1-b)$ probability that the first two shots both miss, and the game continues as if nothing happened in the first two shots. Therefore,

$$(\alpha, \beta) = a(1,0) + (1-a)b(0,1) + (1-a)(1-b)(\alpha, \beta),$$

and hence

$$\begin{cases} \alpha = a + (1-a)(1-b)\alpha = a + a'b'\alpha, \\ \beta = (1-a)b + (1-a)(1-b)\beta = a' + a'b'\beta, \end{cases} \implies \begin{cases} \alpha = \dfrac{a}{1-a'b'}, \\ \beta = \dfrac{a'b}{1-a'b'}. \end{cases}$$

Let the expected number of shots in the contest be e. By the linearity of the expectation, we have

$$e = a \cdot 1 + a'b \cdot 2 + a'b' \cdot (e + 2),$$

where the $(e + 2)$ comes from when Arthur and Boadicea both miss their initial shots (for the 2), and the game continues (for the e), and hence

$$e = \frac{a + 2a'b + 2a'b'}{1 - a'b'} = \frac{a + 2a'}{1 - a'b'} = \frac{2 - a}{1 - a'b'}.$$

On the other hand, we have

$$\frac{\alpha}{a} + \frac{\beta}{b} = \frac{1}{1 - a'b'} + \frac{1 - a}{1 - a'b'} = \frac{2 - a}{1 - a'b'},$$

and therefore

$$e = \frac{\alpha}{a} + \frac{\beta}{b},$$

as desired.