2010.3 Question 1

1. Notice that

$$\begin{aligned}
C &= \frac{1}{n+1}\sum_{k=1}^{n+1} x_k \\
&= \frac{1}{n+1}\left(\sum_{k=1}^{n} x_k + x_{n+1}\right) \\
&= \frac{1}{n+1}\left(nA + x_{n+1}\right).
\end{aligned}$$

2. By expanding the brackets,

$$\begin{aligned}
B &= \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2 \\
&= \frac{1}{n}\sum_{k=1}^{n}(x_k^2 - 2Ax_k + A^2) \\
&= \frac{1}{n}\left[\sum_{k=1}^{n} x_k^2 - 2A\sum_{k=1}^{n} x_k + A^2 n\right] \\
&= \frac{1}{n}\sum_{k=1}^{n} x_k^2 - 2A\frac{1}{n}\sum_{k=1}^{n} x_k + A^2 \\
&= \frac{1}{n}\sum_{k=1}^{n} x_k^2 - 2A^2 + A^2 \\
&= \frac{1}{n}\sum_{k=1}^{n} x_k^2 - A^2.
\end{aligned}$$

3. Similarly, we have

$$D = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k^2 - C^2.$$

Hence,

$$\begin{aligned}
D &= \frac{1}{n+1}\sum_{k=1}^{n+1} x_k^2 - C^2 \\
&= \frac{1}{n+1}\left(\sum_{k=1}^{n} x_k^2 + x_{n+1}^2\right) - \left(\frac{1}{n+1}(nA + x_{n+1})\right)^2 \\
&= \frac{1}{n+1}\left(n(B + A^2) + x_{n+1}^2\right) - \left(\frac{1}{n+1}(nA + x_{n+1})\right)^2 \\
&= \frac{1}{(n+1)^2}\left[(n+1)\left(n(B + A^2) + x_{n+1}^2\right) - (nA + x_{n+1})^2\right] \\
&= \frac{1}{(n+1)^2}\left(nA^2 + n(n+1)B + nx_{n+1}^2 - 2nAx_{n+1}\right) \\
&= \frac{n}{(n+1)^2}\left(A^2 + (n+1)B + x_{n+1}^2 - 2Ax_{n+1}\right) \\
&= \frac{n}{(n+1)^2}\left[\left(A - x_{n+1}\right)^2 + (n+1)B\right]
\end{aligned}$$

Hence,

$$\begin{aligned}(n+1)D - nB &= \frac{n}{n+1}\left[(A - x_{n+1})^2 + (n+1)B\right] - nB \\ &= \frac{n}{n+1} \cdot (A - x_{n+1})^2 + nB - nB \\ &= \frac{n}{n+1} \cdot (A - x_{n+1})^2 \\ &\geq 0,\end{aligned}$$

since a square is always non-negative, and hence

$$(n+1)D \geq nB.$$

On the other hand, notice that

$$\begin{aligned}D - B &= \frac{n}{(n+1)^2}\left[(A - x_{n+1})^2 + (n+1)B\right] - B \\ &= \frac{n}{(n+1)^2}(A - x_{n+1})^2 + \frac{n}{n+1}B - B \\ &= \frac{n}{(n+1)^2}(A - x_{n+1})^2 - \frac{1}{n+1}B,\end{aligned}$$

and hence

$$\begin{aligned}D < B &\iff \frac{n}{(n+1)^2}(A - x_{n+1})^2 - \frac{1}{n+1}B < 0 \\ &\iff \frac{n}{(n+1)^2}(A - x_{n+1})^2 < \frac{1}{n+1}B \\ &\iff (A - x_{n+1})^2 < \frac{n+1}{n}B \\ &\iff -\sqrt{\frac{n+1}{n}B} < A - x_{n+1} < \sqrt{\frac{n+1}{n}B} \\ &\iff -A - \sqrt{\frac{n+1}{n}B} < -x_{n+1} < -A + \sqrt{\frac{n+1}{n}B} \\ &\iff A - \sqrt{\frac{n+1}{n}B} < x_{n+1} < A + \sqrt{\frac{n+1}{n}B},\end{aligned}$$

exactly as desired.