2024 Paper 3

2024.3 Question 1

1. For the first identity, notice that

$$\begin{aligned}\frac{1}{r+1}-\frac{1}{r}+\frac{1}{r^2}&=\frac{r^2-r(r+1)+(r+1)}{r^2(r+1)}\\&=\frac{r^2-r^2-r+r+1}{r^2(r+1)}\\&=\frac{1}{r^2(r+1)},\end{aligned}$$

and hence using this,

$$\begin{aligned}\sum_{r=1}^{N}\frac{1}{r^2(r+1)}&=\sum_{r=1}^{N}\left(\frac{1}{r+1}-\frac{1}{r}+\frac{1}{r^2}\right)\\&=\sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=1}^{N}\frac{1}{r+1}-\sum_{r=1}^{N}\frac{1}{r}\\&=\sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=2}^{N+1}\frac{1}{r}-\sum_{r=1}^{N}\frac{1}{r}\\&=\sum_{r=1}^{N}\frac{1}{r^2}-\frac{1}{1}+\frac{1}{N+1}\\&=\sum_{r=1}^{N}\frac{1}{r^2}-1+\frac{1}{N+1}.\end{aligned}$$

Let $N\to\infty$, and we have $\frac{1}{N+1}\to 0$, and hence

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)}=\sum_{r=1}^{\infty}\frac{1}{r^2}-1=\frac{\pi^2}{6}-1.$$

2. By partial fractions, let

$$\frac{1}{r^2(r+1)(r+2)}=\frac{Ar+B}{r^2}+\frac{C}{r+1}+\frac{D}{r+2}$$

for real constants A, B, C and D.

Hence,

$$(Ar+B)(r+1)(r+2)+Cr^2(r+2)+Dr^2(r+1)=1.$$

Let $r=-2$, we have $D\cdot(-2)^2\cdot(-1)=-4D=1$, and hence $D=-\frac{1}{4}$.
Let $r=-1$, we have $C\cdot(-1)^2\cdot 1=C=1$, and hence $C=1$.
Let $r=0$, we have $B\cdot 1\cdot 2=1$, and hence $B=\frac{1}{2}$.
Considering the coefficient of r^3, we have $A+C+D=0$, and hence $A=-\frac{3}{4}$.
Hence,

$$\frac{1}{r^2(r+1)(r+2)}=-\frac{3}{4}\cdot\frac{1}{r}+\frac{1}{2}\cdot\frac{1}{r^2}+\frac{1}{r+1}-\frac{1}{4}\cdot\frac{1}{r+2}.$$

Therefore,

$$\begin{aligned}
\sum_{r=1}^{N} \frac{1}{r^2(r+1)(r+2)} &= -\frac{3}{4}\sum_{r=1}^{N}\frac{1}{r} + \frac{1}{2}\sum_{r=1}^{N}\frac{1}{r^2} + \sum_{r=1}^{N}\frac{1}{r+1} - \frac{1}{4}\sum_{r=1}^{N}\frac{1}{r+2} \\
&= \frac{1}{2}S_N - \frac{3}{4}\cdot\sum_{r=1}^{N}\frac{1}{r} + \sum_{r=2}^{N+1}\frac{1}{r} - \frac{1}{4}\sum_{r=3}^{N+2}\frac{1}{r} \\
&= \frac{1}{2}S_N - \frac{3}{4}\sum_{r=3}^{N}\frac{1}{r} + \sum_{r=3}^{N}\frac{1}{r} - \frac{1}{4}\sum_{r=3}^{N}\frac{1}{r} \\
&= \frac{1}{2}S_N - \frac{3}{4}\left(\frac{1}{1}+\frac{1}{2}\right) + \left(\frac{1}{2}+\frac{1}{N+1}\right) - \frac{1}{4}\left(\frac{1}{N+1}+\frac{1}{N+2}\right) \\
&= \frac{1}{2}S_N - \frac{9}{8} + \frac{4}{8} + \frac{3}{4}\cdot\frac{1}{N+1} - \frac{1}{4}\cdot\frac{1}{N+2} \\
&= \frac{1}{2}S_N - \frac{5}{8} + \frac{3}{4}\cdot\frac{1}{N+1} - \frac{1}{4}\cdot\frac{1}{N+2}.
\end{aligned}$$

Let $N \to \infty$, we have $\frac{1}{N+1}, \frac{1}{N+2} \to 0$, and hence

$$\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)(r+2)} = \frac{1}{2}\lim_{N\to\infty} S_N - \frac{5}{8} = \frac{\pi^2}{12} - \frac{5}{8}.$$

3. Similarly, let

$$\frac{1}{r^2(r+1)^2} = \frac{A}{r^2} + \frac{B}{r} + \frac{C}{(r+1)^2} + \frac{D}{r+1}$$

for some real constants A, B, C and D.

Hence,

$$1 = A(r+1)^2 + Br(r+1)^2 + Cr^2 + Dr^2(r+1).$$

Let $r = 0$, and we have $A = 1$. Let $r = -1$, and we have $C = 1$. Considering the coefficient of r^3 we have $B + D = 0$, and for r, $2A + B = 0$.

Hence, $B = -2, D = 2$, and

$$\frac{1}{r^2(r+1)^2} = \frac{1}{r^2} - \frac{2}{r} + \frac{1}{(r+1)^2} + \frac{2}{r+1}.$$

Therefore, for natural numbers N, we have

$$\begin{aligned}
\sum_{r=1}^{N} \frac{1}{r^2(r+1)^2} &= \sum_{r=1}^{N}\frac{1}{r^2} + \sum_{r=1}^{N}\frac{1}{(r+1)^2} + 2\sum_{r=1}^{N}\frac{1}{r+1} - 2\sum_{r=1}^{N}\frac{1}{r} \\
&= S_N + \sum_{r=1}^{N+1}\frac{1}{r^2} + 2\sum_{r=2}^{N+1}\frac{1}{r} - 2\sum_{r=1}^{N}\frac{1}{r} \\
&= S_N + S_{N+1} - \frac{1}{1^2} + 2\cdot\frac{1}{N+1} - 2\cdot 1 \\
&= S_N + s_{N+1} + 2\cdot\frac{1}{N+1} - 3.
\end{aligned}$$

Let $N \to \infty$. $S_N, S_{N+1} \to \frac{\pi^2}{6}$, and $\frac{1}{N+1} \to 0$. Hence,

$$\begin{aligned}\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)^2} &= 2 \cdot \frac{\pi^2}{6} - 3 \\ &= \frac{\pi^2}{3} - 3 \\ &= 2 \cdot \left(\frac{\pi^2}{6} - 1\right) - 1 \\ &= 2\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)} - 1 \\ &= \sum_{r=1}^{\infty} \frac{2}{r^2(r+1)} - 1,\end{aligned}$$

as desired.

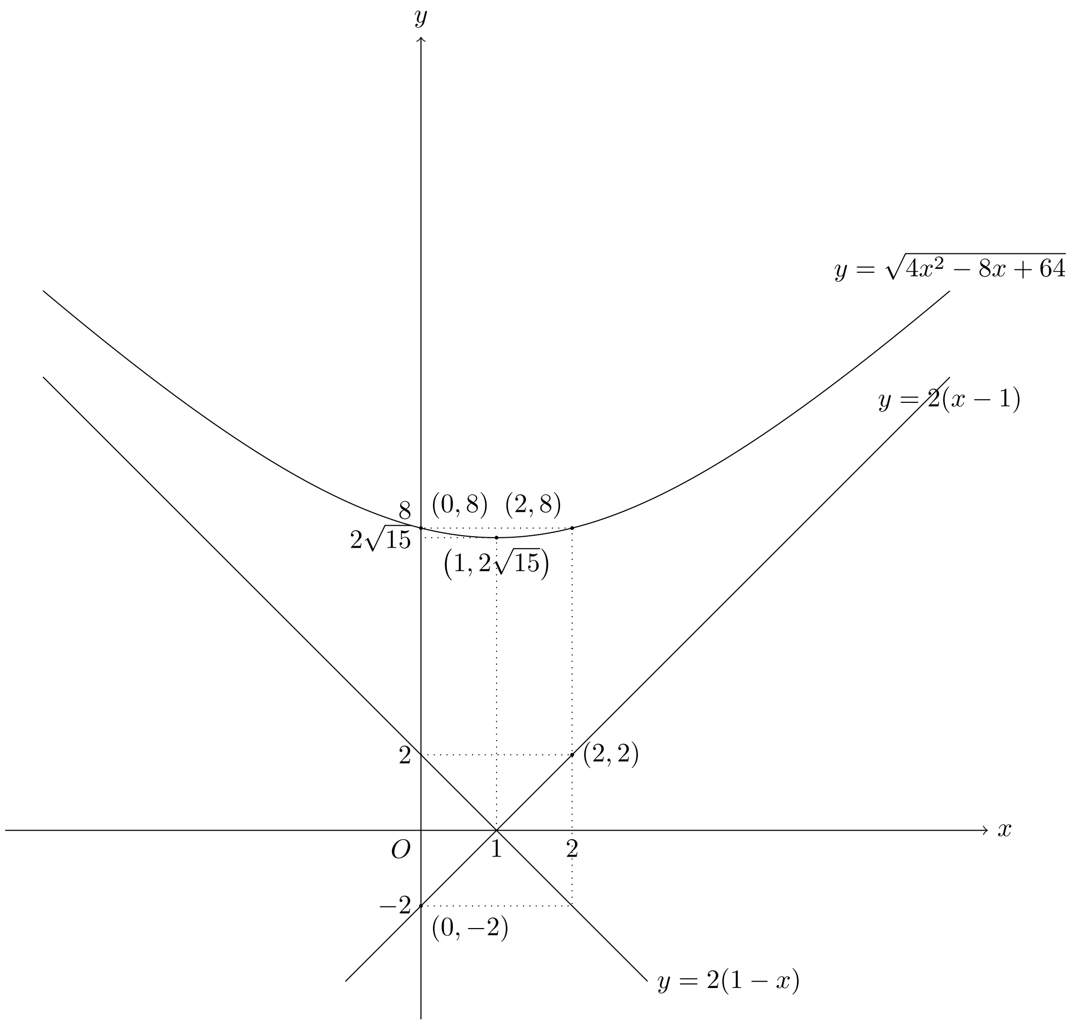
2024.3 Question 2

1. (a) We have
$$\sqrt{4x^2 - 8x + 64} \leq |x+8| \iff 0 \leq 4x^2 - 8x + 64 \leq (x+8)^2.$$
The left inequality can be simplified as follows:
$$\begin{aligned} 4x^2 - 8x + 64 &\geq 0 \\ x^2 - 2x + 16 &\geq 0 \\ (X-1)^2 + 15 &\geq 0, \end{aligned}$$
which is always true.
The right inequality can be simplified as follows:
$$\begin{aligned} 4x^2 - 8x + 64 &\leq (x+8)^2 \\ 4x^2 - 8x + 64 &\leq x^2 + 16x + 64 \\ 3x^2 - 24x &\leq 0 \\ x(x-8) &\leq 0, \end{aligned}$$
which solves to $0 \leq x \leq 8$.
Hence, the solution to the original inequality is $x \in [0, 8]$.

   (b) WE have
$$\sqrt{4x^2 - 8x + 64} \leq |3x-8| \iff 0 \leq 4x^2 - 8x + 64 \leq (3x-8)^2.$$
The left inequality is always true from the previous part.
The right inequality can be simplified as follows:
$$\begin{aligned} 4x^2 - 8x + 64 &\leq (3x-8)^2 \\ 4x^2 - 8x + 64 &\leq 9x^2 - 48x + 64 \\ 5x^2 - 40x &\geq 0 \\ x(x-8) &\geq 0, \end{aligned}$$
which solves to $x \leq 0$ or $x \geq 8$.
Hence, the solution to the original inequality is $x \in (-\infty, 0] \cup [8, \infty)$.

2. (a) We have
$$\begin{aligned} \left(\sqrt{4x^2 - 8x + 64} + 2(x-1)\right) f(x) &= \left(\sqrt{4x^2 - 8x + 64}\right)^2 - [2(x-1)]^2 \\ &= \left(4x^2 - 8x + 64\right) - 4\left(x^2 - 2x + 1\right) \\ &= \left(4x^2 - 8x + 64\right) - \left(4x^2 - 8x + 4\right) \\ &= 60. \end{aligned}$$
Hence,
$$f(x) = \frac{60}{\sqrt{4x^2 - 8x + 64} + 2(x-1)}.$$
As $x \to \infty$, $\sqrt{4x^2 - 8x + 64} \to \infty$, $2(x-1) \to \infty$.
Hence, $f(x) \to 0$ as $x \to \infty$.

   (b) Let $f_1(x) = \sqrt{4x^2 - 8x + 64}$, $f_2(x) = 2(x-1)$.
$f_1(0) = \sqrt{64} = 8$, and $f_2(0) = 2 \cdot (-1) = -2$.
We have $f(x) = f_1(x) - f_2(x) > 0$ from the previous part, and that $f_1(x)$ is defined for all x and is always positive.
Furthermore,
$$f_1(x) = 2\sqrt{x^2 - 2x + 16} = 2\sqrt{(x-1)^2 + 15},$$
and hence f_1 decreases on $(-\infty, 1)$ and increases on $(1, \infty)$, taking a minimum of $f_1(1) = 2\sqrt{15}$.
In terms of symmetry, we have $f_1(1-x) = f_1(1+x)$ and $f_2(1-x) = -f_2(1+x)$. f_2 is an asymptote to f_1 as $x \to \infty$, and $-f_2$ is an asymptote to f_1 as $x \to -\infty$.
Hence, the sketch looks as follows.



3. Let $x = 3$, and we must have $\sqrt{4 \cdot 9 - 5 \cdot 3 + 4} = |3m + c|$, and hence $5 = |3m + c|$.

   This is only achievable for $m = \pm 2$ due to the diagram − the solution set can only be 'one-sided' if on the other side the absolute value is eventually 'parallel' to the curve.

   We let $m = 2$, and hence $5 = |6 + c|$, which gives $c = -1$ or $c = -11$.

   We would like to show that the desired value is $c = -1$, and that $c = -11$ does not work.

$$\sqrt{4x^2 - 5x + 4} \leq |2x - 1| \iff 0 \leq 4x^2 - 5x + 4 \leq (2x-1)^2.$$

   The left inequality can be simplified as

$$0 \leq 4x^2 - 5x + 4 = \left(2x - \frac{5}{4}\right)^2 + \frac{39}{16},$$

   and hence is always true.

   The right inequality can be simplified as

$$\begin{aligned} 4x^2 - 5x + 4 &\leq (2x-1)^2 \\ 4x^2 - 5x + 4 &\leq 4x^2 - 4x + 1 \\ x &\geq 3, \end{aligned}$$

   and hence the solution set to the whole inequality is $x \geq 3$ as desired.

   On the other hand, for the case of $c = -11$, we have

$$\sqrt{4x^2 - 5x + 4} \leq |2x - 11| \iff 0 \leq 4x^2 - 5x + 4 \leq (2x - 11)^2,$$

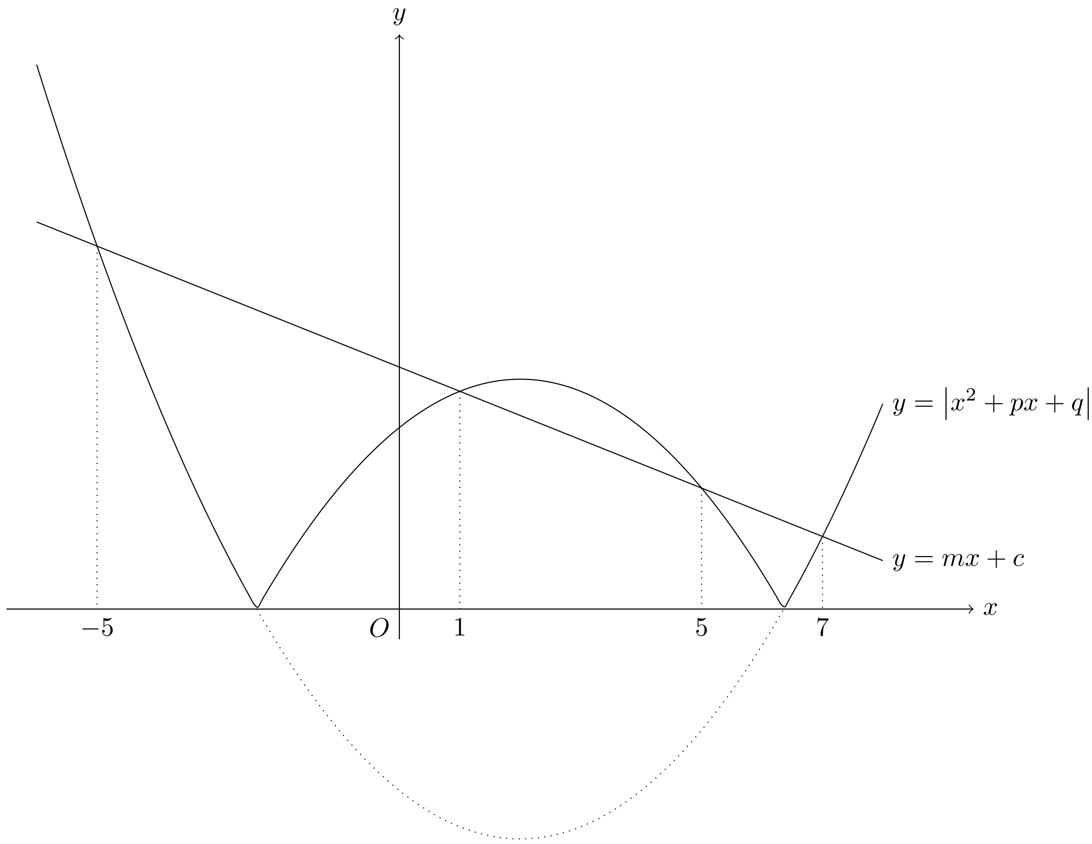
and the left inequality is always true by previously. However, the right inequality simplifies as

$$\begin{aligned} 4x^2 - 5x + 4 &\leq (2x - 11)^2 \\ 4x^2 - 5x + 4 &\leq 4x^2 - 44x + 121 \\ 39x &\leq 117 \\ x &\leq 3, \end{aligned}$$

and the inequality is in the wrong direction.

Hence, a possible value of m is 2, and the corresponding value of c is -1.

4. The diagram as follows shows the only possibility of the configuration.



Hence, we must have $x^2 + px + q = mx + c$ for $x = -5$ and $x = 7$, and $x^2 + px + q = -mx - c$ for $x = 1$ and $x = 5$.

$$\begin{cases} 25 - 5p + q = -5m + c, \\ 49 + 7p + q = 7m + c, \\ \quad 1 + p + q = -(m + c), \\ 25 + 5p + q = -(5m + c). \end{cases}$$

Subtracting the first equation from the final equation gives $10p = -2c$, and hence $c = -5p$.

Subtracting the first equation from the second equation gives us $24 + 12p = 12m$, and hence $m = 2 + p$.

Putting these into the third equation gives

$$\begin{aligned} q &= -m - c-; -1 \\ &= -(2 + p) - (-5p) - p - 1 \\ &= 3p - 3. \end{aligned}$$

Putting all these into the final equation gives

$$\begin{aligned}
25 + 5p + (3p - 3) &= -\left[5(2 + p) + (-5p)\right] \\
25 + 8p - 3 &= -(10 + 5p - 5p) \\
22 + 8p &= -10 \\
8p &= -32 \\
p &= -4,
\end{aligned}$$

and so $q = -15, m = -2, c = 20$. Hence,

$$(p, q, m, c) = (-4, -15, -2, 20).$$

2024.3 Question 3

1. (a) Notice that by partial fractions, we have

$$\frac{x+c}{x(x+1)} = \frac{1-c}{x+1} + \frac{c}{x}.$$

Hence, by differentiating, we have

$$\begin{aligned} g'(x) &= \frac{1}{1+\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) + \frac{1-c}{(x+1)^2} + \frac{c}{x^2} \\ &= -\frac{1}{x^2+x} + \frac{1-c}{(x+1)^2} + \frac{c}{x^2} \\ &= \frac{-x(x+1) + (1-c)x^2 + c(x+1)^2}{(x+1)^2x^2} \\ &= \frac{cx^2 + 2cx + c + x^2 - cx^2 - x^2 - x}{(x+1)^2x^2} \\ &= \frac{(2c-1)x + c}{(x+1)^2x^2}. \end{aligned}$$

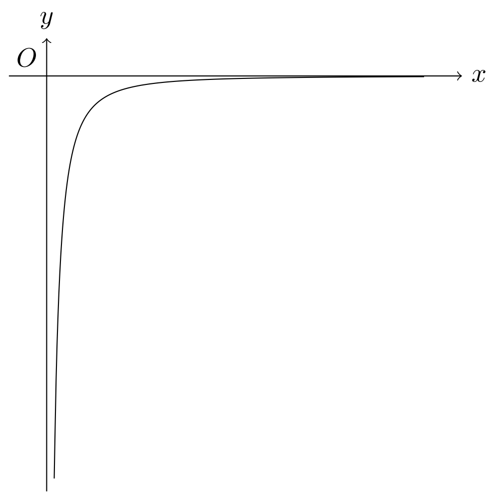
Given that $c \geq \frac{1}{2}$, and $x > 0$, we have $2c - 1 \geq 0$, and $(2c-1)x \geq 0$.
Hence, the numerator satisfies $(2c-1)x + c \geq c \geq \frac{1}{2} > 0$, and the denominator is always positive since is a product of squares, and both squares are non-zero since $x > 0$.
We can now conclude that $g'(x) > 0$ given $c \geq \frac{1}{2}$ for $x > 0$, as desired.

(b) If $0 \leq c < \frac{1}{2}$, $g'(x) < 0$ if and only if

$$\begin{aligned} (2c-1)x + c &< 0 \\ (1-2c)x - c &> 0 \\ (1-2c)x &> c \\ x &> \frac{c}{1-2c}, \end{aligned}$$

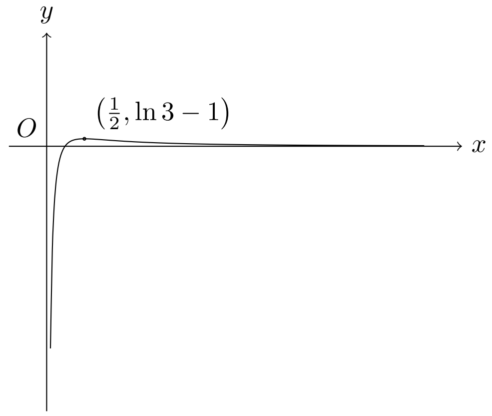
and the values of x are $x > \frac{c}{1-2c}$.

2. (a) If $c = \frac{3}{4} \geq \frac{1}{2}$, we can see that g is always increasing.
As $x \to \infty$, $\frac{x+c}{x(x+1)} \to 0$, $\ln\left(1 + \frac{1}{x}\right) \to \ln 1 = 0$. Hence, $g(x) \to 0$.
Since g is increasing it must stay entirely below the x-axis.
The sketch is as follows.



(b) If $c = \frac{1}{4} \in \left[0, \frac{1}{2}\right)$, it must be the case that $g'(x) > 0$ for $0 < x < \frac{c}{1-2c} = \frac{1}{2}$, and $g'(x) < 0$ for $x > \frac{1}{2}$.

Hence, $x = \frac{1}{2}$ is a maximum on the graph, and the corresponding y-coordinate is $g\left(\frac{1}{2}\right) = \ln 3 - 1$.
Similarly, as $x \to \infty$, $g(x) \to 0$.
The sketch is as follows.



3. We have

$$\begin{aligned}
f(x) &= \left(1 + \frac{1}{x}\right)^{x+c} \\
\ln f(x) &= (x+c)\ln\left(1 + \frac{1}{x}\right) \\
\frac{f'(x)}{f(x)} &= \ln(1+x) - (x+c)\frac{1}{x(x+1)} \\
\frac{f'(x)}{f(x)} &= g(x) \\
f'(x) &= f(x)g(x).
\end{aligned}$$

$f(x)$ is positive for $x > 0$, and hence $f'(x)$ takes the same sign as $g(x)$.

(a) If $c \geq \frac{1}{2}$, g is increasing and has a limit of 0 at infinity. Hence, $g(x)$ is negative for all $x > 0$, which means $f'(x)$ is negative for all $x > 0$, and hence f is decreasing.

(b) If $0 < c < \frac{1}{2}$, g is negative first, then increases to a positive value, and remains positive and approaches 0 decreasing from above. Hence, f' is first positive and then negative, so f must have a turning point.

(c) If $c = 0$,
$$g'(x) = \frac{-x}{(x+1)^2x^2} = -\frac{1}{(x+1)^2x}$$
is always negative, and $\lim_{x\to 0^+} g'(x) = -\infty$, $\lim_{x\to\infty} g'(x) = 0$.
We have
$$g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}.$$
As $x \to 0^+$, $\frac{1}{x} \to \infty$, so $\ln\left(1 + \frac{1}{x}\right) \to \infty$, and $-\frac{1}{x+1} \to -\frac{1}{1} = -1$. Hence, $g(x) \to \infty$.
As $x \to \infty$, $g(x) \to 0$.
Since g is decreasing, it must be the case that g is always positive.
This means that f' is always positive as well, and hence f is increasing.

2024.3 Question 4

1. The angle between a line with gradient m and the positive x-axis is $\arctan m$. Hence, we must have
$$\begin{aligned}\arctan m_1 - \arctan m_2 &= \pm\frac{\pi}{4}\\ \tan\left(\arctan m_1 - \arctan m_2\right) &= \tan\left(\pm\frac{\pi}{4}\right)\\ \frac{m_1 - m_2}{1 + m_1 m_2} &= \pm 1,\end{aligned}$$
as desired.

2. We have $y = \frac{x^2}{4a}$, and hence $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{2a}$. Hence, the tangent to the point $\left(p, \frac{p^2}{4a}\right)$ is given by
$$\begin{aligned}y - \frac{p^2}{4a} &= \frac{p}{2a}(x-p)\\ 4ay - p^2 &= 2p(x-p)\\ 4ay &= 2px - p^2,\end{aligned}$$
with gradient $\frac{2p}{4a} = \frac{p}{2a}$, and the tangent to the point $\left(q, \frac{q^2}{4a}\right)$ is given by $4ay = 2qx + q^2$, with gradient $\frac{q}{2a}$.
Hence, when they intersect, it must be the case that
$$\begin{aligned}2px - p^2 &= 2qx - q^2\\ 2(p-q)x &= p^2 - q^2\\ 2(p-q)x &= (p+q)(p-q)\\ x &= \frac{p+q}{2}\end{aligned}$$
since $p \neq q$.
The y-coordinate is given by
$$\begin{aligned}y &= \frac{2px - p^2}{4a}\\ &= \frac{p^2 + pq - p^2}{4a}\\ &= \frac{pq}{4a}.\end{aligned}$$
If the two curves meet at $\frac{\pi}{4}$, the gradients must satisfy that
$$\begin{aligned}\frac{\frac{p}{2a} - \frac{q}{2a}}{1 + \frac{p}{2a}\cdot\frac{q}{2a}} &= \pm 1\\ \frac{2a(p-q)}{4a^2 + pq} &= \pm 1\\ 2a(p-q) &= \pm\left(4a^2 + pq\right)\\ 4a^2(p-q)^2 &= (4a^2 + pq)^2\\ 4a^2p^2 - 8a^2pq + 4a^2q^2 &= 16a^4 + 8a^2pq + p^2q^2\\ p^2q^2 + 16a^2pq + 16a^4 - 4a^2p^2 - 4a^2q^2 &= 0.\end{aligned}$$
For the intersection of the two tangents, we consider $(y+3a)^2 - (8a^2 + x^2)$.
$$\begin{aligned}(y+3a)^2 - (8a^2 + x^2) &= y^2 + 6ay + 9a^2 - 8a^2 - x^2\\ &= y^2 + 6ay - x^2 + a^2\\ &= \frac{p^2q^2}{16a^2} + 6a\cdot\frac{pq}{4a} - \left(\frac{p+q}{2}\right)^2 + a^2\\ &= \frac{p^2q^2}{16a^2} + \frac{3pq}{2} - \frac{(p+q)^2}{4} + a^2.\end{aligned}$$

We have the following being equivalent:

$$\begin{aligned}(y+3a)^2 &= 8a^2+x^2\\ \frac{p^2q^2}{16a^2}+\frac{3pq}{2}-\frac{(p+q)^2}{4}+a^2 &= 0\\ p^2q^2+3pq\cdot 8a^2-(p+q)^2\cdot 4a^2+a^2\cdot 16a^2 &= 0\\ p^2q^2+24pqa^2-4a^2p^2-4a^2q^2-8pqa^2+16a^4 &= 0\\ p^2q^2+16a^2pq+16a^4-4a^2p^2-4a^2q^2 &= 0,\end{aligned}$$

which was true due to the tangents intersecting at $\frac{\pi}{4}$.

Hence, we must have the intersection of two tangents lie on $(y+3a)^2=8a^2+x^2$, which finishes our proof.

3. Let θ be this acute angle, and from the previous part, we can see that

$$\begin{aligned}4a^2(p-q)^2 &= \tan^2\theta(4a^2+pq)^2\\ 4a^2p^2-8a^2pq+4a^2q^2 &= \tan^2\theta 16a^4+\tan^2\theta 8a^2pq+\tan^2\theta p^2q^2\\ \tan^2\theta p^2q^2+8(\tan^2\theta+1)a^2pq+\tan^2\theta 16a^4 &= 4a^2p^2+4a^2q^2\end{aligned}$$

Given $(y+7a)^2=48a^2+3x^2$ for the intersection of the two tangents, we have

$$\begin{aligned}(y+7a)^2-\left(48a^2+3x^2\right) &= 0\\ \left(\frac{pq}{4a}+7a\right)^2-\left(48a^2+3\left(\frac{p+q}{2}\right)^2\right) &= 0\\ \frac{p^2q^2}{16a^2}+\frac{7pq}{2}+49a^2-48a^2-\frac{3(p+q)^2}{4} &= 0\\ p^2q^2+8a^2\cdot 7pq+16a^4-3(p+q)^2\cdot 4a^2 &= 0\\ p^2q^2+56pqa^2+16a^4-12p^2a^2-12q^2a^2-24pqa^2 &= 0\\ p^2q^2+32pqa^2+16a^4-12p^2a^2-12q^2a^2 &= 0\\ p^2q^2+32pqa^2+16a^4-3\left(\tan^2\theta p^2q^2+8(\tan^2\theta+1)a^2pq+16\tan^2\theta a^4\right) &= 0\\ (1-3\tan^2\theta)p^2q^2+8(1-3\tan^2\theta)pqa^2+16(1-3\tan^2\theta)a^4 &= 0\\ (1-3\tan^2\theta)\left[p^2q^2+8pqa^2+16a^4\right] &= 0\\ (1-3\tan^2\theta)(pq+4a^2)^2 &= 0.\end{aligned}$$

Hence, either $pq+4a^2=0$, or $1-3\tan^2\theta=0$. The former cannot always the case. Therefore, $1-3\tan^2\theta=0$, which gives $\tan\theta=\pm\frac{\sqrt{3}}{3}$.

Since θ is acute, we have $\tan\theta=\frac{\sqrt{3}}{3}$, and hence $\theta=\frac{\pi}{6}$ is the acute angle between the two tangents.

2024.3 Question 5

1. Let
$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \mathbf{N} = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$
and hence we have
$$\operatorname{tr}\mathbf{M} = a + d, \operatorname{tr}\mathbf{N} = e + h.$$
Notice that
$$\mathbf{MN} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}, \mathbf{NM} = \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix},$$
which means
$$\operatorname{tr}(\mathbf{MN}) = ae + bg + cf + dh, \operatorname{tr}(\mathbf{NM}) = ae + cf + bg + dh,$$
and hence $\operatorname{tr}(\mathbf{MN}) = \operatorname{tr}(\mathbf{NM})$ as desired.
We also have
$$\mathbf{M} + \mathbf{N} = \begin{pmatrix} a + e & b + f \\ c + g & d + h \end{pmatrix},$$
meaning $\operatorname{tr}(\mathbf{M} + \mathbf{N}) = a + e + d + h = (a + d) + (e + h) = \operatorname{tr}\mathbf{M} + \operatorname{tr}\mathbf{N}$.

2. We have $\det\mathbf{M} = ad - bc$, and hence
$$\frac{\mathrm{d}}{\mathrm{d}t}\det\mathbf{M} = \dot{a}d + a\dot{d} - \dot{b}c - b\dot{c}.$$
Hence,
$$\text{LHS} = \frac{1}{ad - bd}\left(\dot{a}d + a\dot{d} - \dot{b}c - b\dot{c}\right).$$
On the other hand,
$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \begin{pmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{pmatrix}, \mathbf{M}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$
and hence
$$\begin{aligned}\mathbf{M}^{-1}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} &= \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{pmatrix} \\ &= \frac{1}{ad - bc}\begin{pmatrix} \dot{a}d - b\dot{c} & \dot{b}d - b\dot{d} \\ -\dot{a}c + a\dot{c} & -\dot{b}c + a\dot{d} \end{pmatrix}.\end{aligned}$$
Hence,
$$\begin{aligned}\text{RHS} &= \operatorname{tr}\left(\mathbf{M}^{-1}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}\right) \\ &= \frac{1}{ad - bc}\left(\dot{a}d - b\dot{c} - \dot{b}c + a\dot{d}\right) \\ &= \frac{1}{ad - bc}\left(\dot{a}d + a\dot{d} - b\dot{c} - \dot{b}c\right) \\ &= \text{LHS},\end{aligned}$$
as desired.

3. $\det\mathbf{M} \neq 0$ since $\mathbf{M}$ is non-singular, and hence left-multiplying by $\mathbf{M}^{-1}$ on both sides gives us
$$\mathbf{M}^{-1}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \mathbf{N} - \mathbf{M}^{-1}\mathbf{NM}.$$

Taking trace on both sides, we have

$$\begin{aligned}\frac{1}{\det\mathbf{M}}\frac{\mathrm{d}}{\mathrm{d}t}\det\mathbf{M} &= \operatorname{tr}\left(\mathbf{M}^{-1}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}\right)\\ &= \operatorname{tr}\left(\mathbf{N}-\mathbf{M}^{-1}\mathbf{N}\mathbf{M}\right)\\ &= \operatorname{tr}\mathbf{N}-\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{N}\mathbf{M}\right)\\ &= \operatorname{tr}\mathbf{N}-\operatorname{tr}\left(\left(\mathbf{M}^{-1}\mathbf{N}\right)\mathbf{M}\right)\\ &= \operatorname{tr}\mathbf{N}-\operatorname{tr}\left(\mathbf{M}\left(\mathbf{M}^{-1}\mathbf{N}\right)\right)\\ &= \operatorname{tr}\mathbf{N}-\operatorname{tr}\left(\left(\mathbf{M}\mathbf{M}^{-1}\right)\mathbf{N}\right)\\ &= \operatorname{tr}\mathbf{N}-\operatorname{tr}\left(\mathbf{I}\mathbf{N}\right)\\ &= \operatorname{tr}\mathbf{N}-\operatorname{tr}\mathbf{N}\\ &= 0.\end{aligned}$$

Hence, $\frac{\mathrm{d}}{\mathrm{d}t}\det\mathbf{M}=0$, which means $\det\mathbf{M}$ is a constant independent of t.
Directly taking trace on both sides, we have

$$\begin{aligned}\operatorname{tr}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} &= \operatorname{tr}(\mathbf{MN}-\mathbf{NM})\\ &= \operatorname{tr}(\mathbf{MN})-\operatorname{tr}(\mathbf{NM})\\ &= 0,\end{aligned}$$

and note

$$\operatorname{tr}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}\mathbf{M},$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}\mathbf{M}=0,$$

meaning $\operatorname{tr}\mathbf{M}$ is a constant independent of t.
Notice that

$$\operatorname{tr}\left(\mathbf{M}^2\right)=\operatorname{tr}\left[\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right]=a^2+bc+bc+d^2=a^2+2bc+d^2.$$

Since $\operatorname{tr}\mathbf{M}$ and $\det\mathbf{M}$ are both independent of t, we must have

$$\begin{aligned}(\operatorname{tr}\mathbf{M})^2-2\det\mathbf{M} &= (a+d)^2-2(ad-bc)\\ &= a^2+2ad+d^2-2ad+2bc\\ &= a^2+2bc+d^2\\ &= \operatorname{tr}\left(\mathbf{M}^2\right)\end{aligned}$$

is independent of t as well.
Let

$$\mathbf{M}=\begin{pmatrix}A+x & b\\ c & D-x\end{pmatrix},$$

the diagonal ones being so since the trace is independent of t. Here, x is a function of t.
By differentiating,

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}=\begin{pmatrix}\dot{x} & \dot{b}\\ \dot{c} & -\dot{x}\end{pmatrix},$$

and the right-hand side satisfies

$$\begin{aligned}\mathbf{MN}-\mathbf{NM} &= \begin{pmatrix}A+x & b\\ c & D-x\end{pmatrix}\begin{pmatrix}t & t\\ & t\end{pmatrix}-\begin{pmatrix}t & t\\ & t\end{pmatrix}\begin{pmatrix}A+x & b\\ c & D-x\end{pmatrix}\\ &= \begin{pmatrix}t(A+x) & (A+x)t+bt\\ ct & ct+(D-x)t\end{pmatrix}-\begin{pmatrix}t(A+x)+ct & bt+t(D-x)\\ ct & t(D-x)\end{pmatrix}\\ &= \begin{pmatrix}-ct & (A-D+2x)t\\ 0 & ct.\end{pmatrix}\end{aligned}$$

Comparing the components, we see that $\dot{c} = 0$, meaning that c is a constant: $c = C$.
Hence, $\dot{x} = -Ct$, which solves to $x = -\frac{Ct^2}{2}$, since $x = 0$ when $t = 0$.
This means

$$\dot{b} = (A - D + 2x)t = (A - D - Ct^2)t,$$

and hence

$$b = \frac{(A-D)t^2}{2} - \frac{Ct^4}{4} + B$$

since $b = B$ when $t = 0$.
Hence,

$$\mathbf{M} = \begin{pmatrix} A - Ct^2/2 & (A-D)t^2/2 - Ct^4/4 \\ C & D + Ct^2/2 \end{pmatrix}$$

is the solution given the conditions.

4. By rearranging, we have

$$\mathbf{N} = \mathbf{M}^{-1} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}.$$

Hence, let

$$\mathbf{M} = \begin{pmatrix} 1 + e^t & \\ & 1 - e^t \end{pmatrix},$$

we have

$$\operatorname{tr} \mathbf{M} = 2$$

which is non-zero and independent of t.
Hence,

$$\mathbf{M}^{-1} = \frac{1}{1 - e^{2t}} \begin{pmatrix} 1 - e^t & \\ & 1 + e^t \end{pmatrix}, \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \begin{pmatrix} e^t & \\ & -e^t \end{pmatrix},$$

so

$$\begin{aligned}\mathbf{N} &= \frac{1}{1 - e^{2t}} \begin{pmatrix} 1 - e^t & \\ & 1 + e^t \end{pmatrix} \begin{pmatrix} e^t & \\ & -e^t \end{pmatrix} \\ &= \frac{1}{1 - e^{2t}} \begin{pmatrix} e^t(1 - e^t) & \\ & -e^t(1 + e^t) \end{pmatrix},\end{aligned}$$

which gives

$$\operatorname{tr} \mathbf{N} = \frac{e^{2t}}{e^{2t} - 1}$$

which is clearly non-zero.

2024.3 Question 6

1. (a) We have

$$\begin{aligned}\frac{\mathrm{d}x-y}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t}-\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= (-x+3y+u)-(x+y+u)\\ &= -2x+2y\\ &= -2(x-y).\end{aligned}$$

This is a differential equation for $x-y$ in terms of t, and hence it solves to

$$x-y=Ae^{-2t}.$$

If $x=y=0$ for some $t>0$, then it must be the case that $A=0$, giving $x-y=0$, and $x=y$. Therefore, for $t=0$, we must also necessarily have $x_0=y_0$.

(b) Given that $x_0=y_0$, we must have $x=y$ for all $t>0$. Hence,

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -x+3x+u\\ \frac{\mathrm{d}x}{\mathrm{d}t} &= 2x+u\\ \frac{\mathrm{d}x}{2x+u} &= \mathrm{d}t\\ \ln|2x+u| &= 2t+C\\ 2x+u &= Ae^{2t}.\end{aligned}$$

Since at $t=0$, $x=x_0$, we must have $A=2x_0+u$, and hence

$$2x+u=(2x_0+u)e^{2t},$$

and rearranging gives

$$u=\frac{2(x_0e^{2t}-x)}{1-e^{2t}}.$$

The particle is at origin at time $t=T>0$, and hence $x=y=0$ for $t=T$, and hence

$$u=\frac{2x_0e^{2T}}{1-e^{2T}}.$$

This ensures the particle is at origin as well since this ensures the particle is at $x=0$ for $t=T$, and $y=x$ so $y=0$ as well.

2. (a) Consider $\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}z}{\mathrm{d}t}-2\frac{\mathrm{d}y}{\mathrm{d}t}$, and we have

$$\begin{aligned}\frac{\mathrm{d}x+z-2y}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}z}{\mathrm{d}t}-2\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= (4y-5z+u)+(x-2y+u)-2(x-2z+u)\\ &= 4y-5z+u+x-2y+u-2x+4z-2u\\ &= -x-z+2y,\end{aligned}$$

and hence

$$x+z-2y=Ae^{-t}.$$

Since the particle is at the origin at some time $t>0$, we must have $A=0$, and hence

$$x+z-2y=0,$$

which means $y=\frac{x+z}{2}$ for all time t.
At time $t=0$, $y_0=\frac{x_0+z_0}{2}$, and so y_0 is the mean of x_0 and z_0.

(b) Since $2y = x + z$, we must have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2(x+z) - 5z + u = 2x - 3z + u,$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = x - (x+z) + u = -z + u.$$

Hence, considering $\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}z}{\mathrm{d}t}$, we have

$$\begin{aligned}\frac{\mathrm{d}x - z}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}z}{\mathrm{d}t} \\ &= (2x - 3z + u) - (-z + u) \\ &= 2(x - z),\end{aligned}$$

which gives

$$x - z = Ae^{2t}.$$

Since the particle is at the origin for some $t > 0$, we must have $A = 0$. This means $x = z$ for all t, and further we have $x = y = z$ for all t since $2y = x + z$.

At $t = 0$, this means $x_0 = y_0 = z_0$ as desired.

(c) Given that $x_0 = y_0 = z_0$, all previous parts still apply, since the boundary condition of $2y = x + z$ and $x = z$ holds for $t = 0$. Hence, $x = y = z$ for all t, and

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -x + u \\ \frac{\mathrm{d}x}{x - u} &= -\,\mathrm{d}t \\ \ln|x - u| &= -t + C \\ x - u &= Ae^{-t}.\end{aligned}$$

At $t = 0$, $x = x_0$, we must have $A = x_0 - u$, and hence

$$x - u = (x_0 - u)e^{-t},$$

and rearranging gives

$$u = \frac{x_0 e^{-t} - x}{1 - e^{-t}}.$$

The particle is at origin at a time $t = T > 0$, and hence $x = y = z = 0$ for $t = T$, and hence

$$u = \frac{x_0 e^{-T}}{1 - e^{-T}} = \frac{x_0}{1 + e^{T}}.$$

This ensures the particle is at origin as well since this ensures the particle is at $x = 0$ for $t = T$, and $x = y = z$, so $y = z = 0$ as well.

2024.3 Question 7

1. For the left inequality, $f(n) > 0$ since $f(n) > \frac{1}{n+1} > 0$.

   For the right inequality, we notice that

$$\begin{aligned}
f(n) &= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \\
&< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots \\
&= \frac{1}{n+1} \cdot \frac{1}{1 - \frac{1}{n+1}} \\
&= \frac{1}{(n+1) - 1} \\
&= \frac{1}{n}.
\end{aligned}$$

   Hence,

$$0 < f(n) < \frac{1}{n}.$$

2. For the left inequality, by grouping consecutive terms, we have

$$\begin{aligned}
g(n) &= \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} \\
&\quad + \frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)} + \cdots \\
&= \left(\frac{1}{n+1} - \frac{1}{(n+1)(n+2)}\right) \\
&\quad + \left(\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)}\right) + \cdots \\
&> \left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\
&\quad + \left(\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)}\right) + \cdots \\
&= 0 + 0 + \cdots \\
&= 0,
\end{aligned}$$

   using the inequality

$$\frac{1}{(n+1)\cdots(n+k)} > \frac{1}{(n+1)\cdots(n+k)(n+k+1)}.$$

For the right inequality, by grouping consecutive after the first one, we have

$$\begin{aligned} g(n) &= \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} \\ &\quad - \frac{1}{(n+1)(n+2)(n+3)(n+4)} + \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} - \cdots \\ &= \frac{1}{n+1} - \left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}\right) \\ &\quad - \left(\frac{1}{(n+1)(n+2)(n+3)(n+4)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}\right) - \cdots \\ &< \frac{1}{n+1} - \left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)}\right) \\ &\quad - \left(\frac{1}{(n+1)(n+2)(n+3)(n+4)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)}\right) - \cdots \\ &= \frac{1}{n+1} - 0 - 0 - \cdots \\ &= \frac{1}{n+1}, \end{aligned}$$

using the inequality

$$\frac{1}{(n+1)\cdots(n+k-1)(n+k)} < \frac{1}{(n+1)\cdots(n+k-1)}.$$

Hence,

$$0 < g(n) < \frac{1}{n+1}.$$

3. The infinite series for e is given by

$$e = \sum_{t=0}^{\infty} \frac{1}{t!},$$

and notice that

$$f(n) = \sum_{t=1}^{\infty} \frac{n!}{(n+t)!} = n! \sum_{t=1}^{\infty} \frac{1}{(n+t)!}.$$

Hence,

$$\begin{aligned} (2n)!e - f(2n) &= (2n)! \sum_{t=0}^{\infty} \frac{1}{t!} - (2n)! \sum_{t=1}^{\infty} \frac{1}{(2n+t)!} \\ &= (2n)! \left(\sum_{t=0}^{\infty} \frac{1}{t!} - \sum_{t=2n+1}^{\infty} \frac{1}{t!}\right) \\ &= (2n)! \sum_{t=0}^{2n} \frac{1}{t!} \\ &= \sum_{t=0}^{2n} \frac{(2n)!}{t!}. \end{aligned}$$

Since $t \leq 2n$, the terms in the sum represents the number of ways to arrange $(2n-t)$ items out of $2n$ items, which must be integers. Hence, the sum is an integer as well.

Similarly, the infinite series for e^{-1} is given by

$$e^{-1} = \sum_{t=0}^{\infty} \frac{(-1)^t}{t!},$$

and notice that

$$g(n) = -\sum_{t=1}^{\infty} \frac{(-1)^t n!}{(n+t)!} = -n! \sum_{t=1}^{\infty} \frac{(-1)^t}{(n+t)!}.$$

Hence,

$$\begin{aligned}
\frac{(2n)!}{e} + g(2n) &= (2n)! \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} - (2n)! \sum_{t=1}^{\infty} \frac{(-1)^t}{(n+t)!} \\
&= (2n)! \left(\sum_{t=0}^{\infty} \frac{(-1)^t}{t!} - \sum_{t=2n+1}^{\infty} \frac{(-1)^t}{t!} \right) \\
&= (2n)! \sum_{t=0}^{2n} \frac{(-1)^t}{t!} \\
&= \sum_{t=0}^{2n} \frac{(-1)^t (2n)!}{t!},
\end{aligned}$$

and by the same argument, since $t \leq 2n$, this must be an integer as well.

4. By the previous part, let $a(n) = f(2n) - (2n)!e$, and $b(n) = g(2n) + \frac{(2n)!}{e}$, we must have that $a, b : \mathbb{N} \to \mathbb{Z}$ since they are integers.

   Using this notation,

$$\begin{aligned}
qf(2n) + pg(2n) &= qa(2n) + qe(2n)! + pb(2n) - \frac{p}{e}(2n)! \\
&= qa(2n) + pb(2n) + \left(qe - \frac{p}{e} \right) (2n)! \\
&= qa(2n) + pb(2n)
\end{aligned}$$

   must be an integer, since $p, q, a(2n), b(2n)$ are all integers.

5. Assume B.W.O.C. that e^2 is irrational. Then there exists natural numbers p, q such that

$$e^2 = \frac{p}{q} \iff qe = \frac{p}{e}.$$

   Since $e^2 > 1$, $p > q$.

   On one hand, we have $qf(2n) + pg(2n) > 0$.

   On the other hand, let $n = p$,

$$\begin{aligned}
qf(2n) + pg(2n) &< q \cdot \frac{1}{2p} + p \cdot \frac{1}{2p+1} \\
&< q \cdot \frac{1}{2p} + p \cdot \frac{1}{2p} \\
&= \frac{p+q}{2p} \\
&< \frac{2p}{2p} \\
&= 1.
\end{aligned}$$

   This means

$$0 < qf(2p) + pg(2p) < 1.$$

   But by the previous part, $qf(2n) + pg(2n)$ is an integer for all positive integer n, and $n = p$ is a positive integer. This leads to a contradiction.

   Hence, such p and q does not exist, meaning e^2 is not rational, hence e^2 is irrational.

2024.3 Question 8

1. $(y-x+3)(y+x-5)=0$ if and only if $y-x+3=0$, or $y+x-5=0$. In the first case, $y=x-3$, representing a straight line with gradient 1, and in the second case, $y=-x+5$, representing a straight line with gradient -1.

   The equation represents a pair of straight lines with gradients 1 and -1 if and only if it could be factorised into the form $(y-x+a)(y+x-b)$.

$$\begin{aligned}(y-x+a)(y+x+b)&=y^2+xy+by-xy-x^2-bx+ay+ax+ab\\&=y^2-x^2+(a+b)y+(a-b)x+ab,\end{aligned}$$

   and $p=a+b, q=a-b, r=ab$.

   On one hand, if it could be factorised into this form, we have

$$p^2-q^2=(a+b)^2-(a-b)^2=a^2+2ab+b^2-a^2+2ab-b^2=4ab=4r.$$

   On the other hand, let $a=\frac{p+q}{2}$, $b=\frac{p-q}{2}$, and we have

$$a+b=p, a-b=q, ab=\frac{p+q}{2}\frac{p-q}{2}=\frac{p^2-q^2}{4}=\frac{4r}{4}=r.$$

   This shows that this is a necessary and sufficient condition, which finishes our proof.

2. Since the point (x,y) lies on C_1, we must have $y=x^2$, and $y-x^2=0$.

   Since it lies on C_2, we must have $x=y^2+2sy+s(s+1)$, and $y^2+2sy+s(s+1)-x$.

   Hence,

$$\begin{aligned}\text{LHS}&=y^2+2sy+s(s+1)-x+k(y-x^2)\\&=0+k\cdot 0\\&=0\\&=\text{RHS}\end{aligned}$$

   for any real number k.

   Let $k=1$, by rearranging, we have

$$y^2-x^2+(2s+1)y-x+s(s+1)=0.$$

   We notice that

$$\begin{aligned}(2s+1)^2-(-1)^2&=4s^2+4s+1-1\\&=4s^2+4s\\&=4s(s+1),\end{aligned}$$

   which means that this represents a pair of straight lines with gradients 1 and -1. The four points of intersection must lie on them.

3. By part (ii), we have $a=\frac{(2s+1)-1}{2}=s$, and $b=\frac{(2s+1)-(-1)}{2}=s+1$. This means

$$(y-x+s)(y+x+s+1)=0,$$

   and the lines are $y=x-s$ and $y=-x-s-1$.

   Since a straight line may at most meet a polynomial twice, we must have $y=x-s$ meets $y=x^2$ at two distinct point, and $y=-x-s-1$ meets $y=x^2$ at two distinct points as well.

   $x^2=x-s \iff x^2-x+s=0$, and hence $1-4s>0$, which shows that $s<\frac{1}{4}$.

   $x^2--x-s-1 \iff x^2+x+(s+1)=0$, and hence $1-4(s+1)>0$, which shows that $s<-\frac{3}{4}$.

   Hence, $s<-\frac{3}{4}$.

4. The lines are $y = x - s$ and $y = -x - s - 1$. Since $s < -\frac{3}{4}$, both lines intersect $y = x^2$ on precisely two points, since the discriminant for the quadratic is positive. Hence, we just have to show that none of those four points are the same.

   This could only be the case of the intersection of the intersection of the two lines, which is $\left(-\frac{1}{2}, -\frac{2s+1}{2}\right)$. This lies on $y = x^2$ if and only if

$$-\frac{2s+1}{2} = \left(-\frac{1}{2}\right)^2 \iff -s - \frac{1}{2} = \frac{1}{4} \iff s = -\frac{3}{4}$$

   which is not the case here.

   Hence, C_1 and C_2 must intersect at four distinct points.

2024.3 Question 11

1. We notice that

$$\begin{aligned}\text{LHS} &= r\binom{2n}{r} \\ &= r \cdot \frac{(2n)!}{r!(2n-r)!} \\ &= \frac{(2n)!}{(r-1)!(2n-r)!},\end{aligned}$$

   and

$$\begin{aligned}\text{RHS} &= (2n+1-r)\binom{2n}{2n+1-r} \\ &= (2n+1-r) \cdot \frac{(2n)!}{(r-1)!(2n+1-r)!} \\ &= \frac{(2n)!}{(r-1)!(2n-r)!}.\end{aligned}$$

   Hence,

$$r\binom{2n}{r} = (2n+1-r)\binom{2n}{2n+1-r}$$

   as desired.

   Summing this from $r = n+1$ to $2n$, we have

$$\begin{aligned}\sum_{r=n+1}^{2n} r\binom{2n}{r} &= \sum_{r=n+1}^{2n} (2n+1-r)\binom{2n}{2n+1-r} \\ &= \sum_{r=1}^{n}(2n+1-(2n+1-r))\binom{2n}{2n+1-(2n+1-r)} \\ &= \sum_{r=1}^{n} r\binom{2n}{r},\end{aligned}$$

   and hence

$$\begin{aligned}\sum_{r=0}^{2n} r\binom{2n}{r} &= \sum_{r=1}^{2n} r\binom{2n}{r} \\ &= \sum_{r=1}^{n} r\binom{2n}{r} + \sum_{r=n+1}^{2n} r\binom{2n}{r} \\ &= \sum_{r=n+1}^{2n} r\binom{2n}{r} + \sum_{r=n+1}^{2n} r\binom{2n}{r} \\ &= 2\sum_{r=n+1}^{2n} r\binom{2n}{r},\end{aligned}$$

   as desired.

2. For $n+1 \leq x \leq 2n$, we have

$$\mathrm{P}(X = x) = 2 \cdot \frac{\binom{2n}{x}}{2^{2n}}.$$

   For $x = n$, we have

$$\mathrm{P}(X = x) = \frac{\binom{2n}{n}}{2^{2n}}.$$

We have $n \leq X \leq 2n$, and hence

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{x=n}^{2n} x \,\mathrm{P}(X = x) \\
&= \frac{n\binom{2n}{n}}{2^{2n}} + \frac{2}{2^{2n}} \sum_{x=n+1}^{2n} x \binom{2n}{x} \\
&= \frac{n\binom{2n}{n}}{2^{2n}} + 2^{-2n} \sum_{r=0}^{2n} r \binom{2n}{r} \\
&= \frac{n\binom{2n}{n}}{2^{2n}} + 2^{-2n}(2n)2^{2n-1} \\
&= n + \frac{n\binom{2n}{n}}{2^{2n}} \\
&= n\left(1 + \frac{1}{2^{2n}}\binom{2n}{n}\right)
\end{aligned}$$

as desired.

3. First, we have that

$$\frac{1}{2^{2n}}\binom{2n}{n} > 0$$

for all positive integers n.

Taking the ratio of two consecutive terms, we have

$$\begin{aligned}
\frac{\frac{1}{2^{2n}}\binom{2n}{n}}{\frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1}} &= \frac{2^{2n+2}\frac{(2n)!}{n!n!}}{2^{2n}\frac{(2n+2)!}{(n+1)!(n+1)!}} \\
&= 4 \cdot \frac{(n+1)^2}{(2n+2)(2n+1)}.
\end{aligned}$$

We have that the following are equivalent:

$$\begin{aligned}
\frac{1}{2^{2n}}\binom{2n}{n} &> \frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1} \\
\frac{\frac{1}{2^{2n}}\binom{2n}{n}}{\frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1}} &> 1 \\
\frac{4(n+1)^2}{(2n+2)(2n+1)} &> 1 \\
4n^2 + 8n + 4 &> 4n^2 + 6n + 2 \\
2n + 2 &> 0
\end{aligned}$$

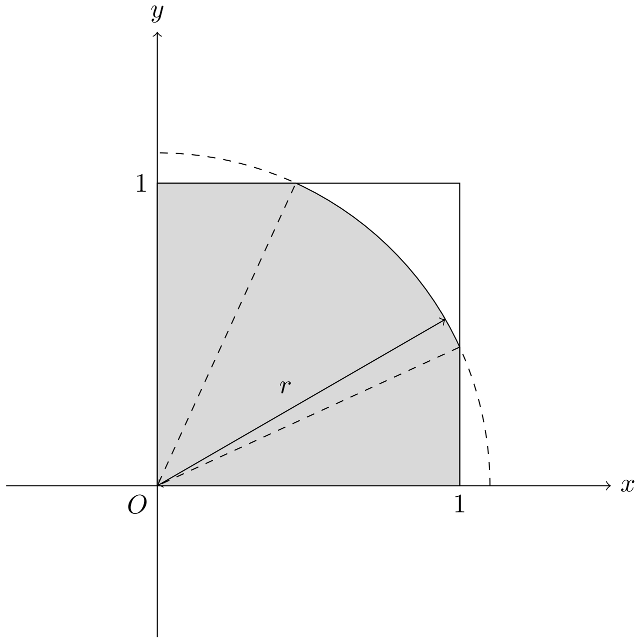
and this obviously true for all positive integers n.

This means that $\frac{1}{2^{2n}}\binom{2n}{n}$ decreases as n increases.

4. The winning is given by $X - n$, and hence the expected winnings per pound is $\frac{1}{2^{2n}}\binom{2n}{n}$. This is maximised when $n = 1$ which gives a value of $\frac{1}{2}$.

2024.3 Question 12

1. For $1 \leq r \leq \sqrt{2}$, the diagram looks as follows.



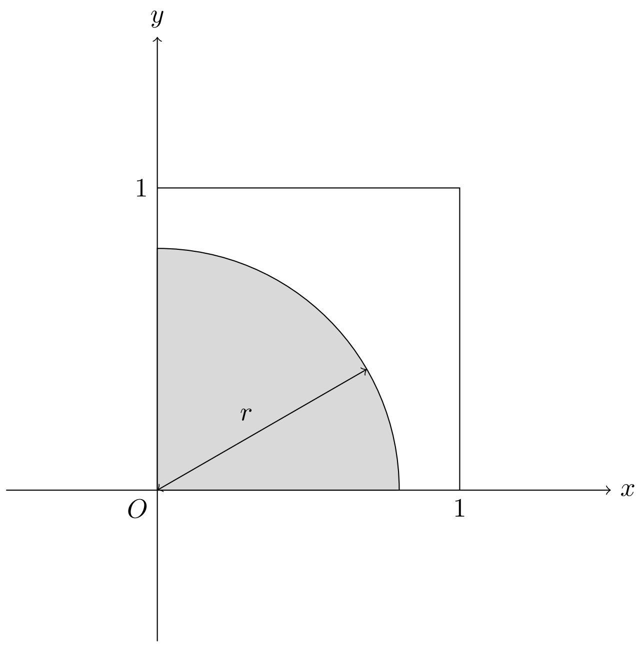
The angle between the (shallower) radius which just intersects the square and x axis is given by $\arccos \frac{1}{r}$, and so is the one steeper and the y-axis.

Hence, the cumulative distribution function is given by

$$\begin{aligned}
\mathrm{P}(R \leq r) &= \frac{\text{shaded area}}{1^2} \\
&= \text{shaded area} \\
&= \frac{1}{2} \cdot r^2 \cdot \left(\frac{\pi}{2} - 2\arccos\frac{1}{r}\right) + 2 \cdot \frac{1}{2} \cdot 1 \cdot \sqrt{r^2 - 1} \\
&= \sqrt{r^2 - 1} + \frac{\pi r^2}{4} - r^2 \arccos\frac{1}{r},
\end{aligned}$$

as desired.

For $0 \leq r \leq 1$, the diagram is as follows.



Hence,

$$P(R \leq r) = \text{shaded area} = \frac{\pi r^2}{4}.$$

Hence, the cumulative distribution function is given by

$$P(R \leq r) = \begin{cases} 0, & r < 0, \\ \frac{\pi r^2}{4}, & 0 \leq r < 1, \\ \sqrt{r^2 - 1} + \frac{\pi r^2}{4} - r^2 \arccos \frac{1}{r}, & 1 \leq r < 2, \\ 1, & 2 \leq r. \end{cases}$$

2. Let f be the probability density function of R. Hence, by differentiating, for $0 \leq r \leq \sqrt{2}$, it is given by

$$\begin{aligned} f(r) &= \frac{\mathrm{d}}{\mathrm{d}r} P(R \leq r) \\ &= \begin{cases} \frac{\pi r}{2}, & 0 \leq r \leq 1, \\ \frac{r}{\sqrt{r^2-1}} + \frac{\pi r}{2} - 2r \arccos \frac{1}{r} - \frac{1}{\sqrt{1-\left(\frac{1}{r}\right)^2}}, & 1 \leq r \leq \sqrt{2}, \end{cases} \\ &= \begin{cases} \frac{\pi r}{2}, & 0 \leq r \leq 1, \\ \frac{\pi r}{2} - 2r \arccos \frac{1}{r}, & 1 \leq r \leq \sqrt{2}. \end{cases} \end{aligned}$$

Hence, the expectation is given by

$$\begin{aligned}
\mathrm{E}(R) &= \int_0^1 r \cdot \frac{\pi r}{2}\,\mathrm{d}r + \int_1^{\sqrt{2}} r \cdot \left[\frac{\pi r}{2} - 2r\arccos\frac{1}{r}\right]\mathrm{d}r \\
&= \int_0^{\sqrt{2}} \frac{\pi r^2}{2}\,\mathrm{d}r - 2\int_1^{\sqrt{2}} r^2 \arccos\frac{1}{r}\,\mathrm{d}r \\
&= \left[\frac{\pi r^3}{6}\right]_0^{\sqrt{2}} - \frac{2}{3}\int_1^{\sqrt{2}} \arccos\frac{1}{r}\,\mathrm{d}r^3 \\
&= \frac{2\sqrt{2}\pi}{6} - \frac{2}{3}\left[\arccos\frac{1}{r}\cdot r^3\right]_1^{\sqrt{2}} + \frac{2}{3}\int_1^{\sqrt{2}} r^3\,\mathrm{d}\arccos\frac{1}{r} \\
&= \frac{\sqrt{2}\pi}{3} - \frac{2}{3}\cdot\arccos\frac{1}{\sqrt{2}}\cdot 2\sqrt{2} + \frac{2}{3}\cdot\arccos 1\cdot 1 + \frac{2}{3}\cdot\int_1^{\sqrt{2}} r^3\cdot\left(-\frac{1}{r^2}\right)\cdot\left(-\frac{1}{\sqrt{1-\left(\frac{1}{r}\right)^2}}\right)\mathrm{d}r \\
&= \frac{\sqrt{2}\pi}{3} - \frac{2}{3}\cdot\frac{\pi}{4}\cdot 2\sqrt{2} + \frac{2}{3}\int_1^{\sqrt{2}} r\cdot\frac{r}{\sqrt{r^2-1}}\,\mathrm{d}r \\
&= \frac{\sqrt{2}\pi}{3} - \frac{\sqrt{2}\pi}{3} + \frac{2}{3}\int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2-1}}\,\mathrm{d}r \\
&= \frac{2}{3}\int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2-1}}\,\mathrm{d}r,
\end{aligned}$$

as desired.

3. To integrate this, we let $r = \cosh t$, and hence $\frac{\mathrm{d}r}{\mathrm{d}t} = \sinh t$. When $r = 1$, $t = 0$. When $r = \sqrt{2}$, $t = \ln\left(\sqrt{2} + \sqrt{\sqrt{2}^2 - 1}\right) = \ln(\sqrt{2}+1)$.

   Hence,

$$\begin{aligned}
\mathrm{E}(R) &= \frac{2}{3}\int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2-1}}\,\mathrm{d}r \\
&= \frac{2}{3}\int_0^{\ln(\sqrt{2}+1)} \frac{\cosh^2 t}{\sinh t}\cdot\sinh t\,\mathrm{d}t \\
&= \frac{2}{3}\int_0^{\ln(\sqrt{2}+1)} \cosh^2 t\,\mathrm{d}t \\
&= \frac{2}{3}\int_0^{\ln(\sqrt{2}+1)} \frac{e^{2t}+e^{-2t}+2}{4}\,\mathrm{d}t \\
&= \frac{1}{2}\left[e^{2t} - e^{-2t}\right]_0^{\ln(\sqrt{2}+1)} + \frac{1}{3}\left[t\right]_0^{\ln(\sqrt{2}+1)} \\
&= \frac{1}{12}\cdot\left[(\sqrt{2}+1)^2 - (\sqrt{2}+1)^{-2} - e^{2\cdot 0} + e^{-2\cdot 0}\right] + \frac{1}{3}\cdot\left(\ln(\sqrt{2}+1) - 0\right) \\
&= \frac{1}{2}\left[2+1+2\sqrt{2} - (\sqrt{2}-1)^2\right] + \frac{1}{3}\ln(\sqrt{2}+1) \\
&= \frac{1}{2}\cdot 4\sqrt{2} + \frac{1}{3}\ln(\sqrt{2}+1) \\
&= \frac{1}{3}\left(\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right),
\end{aligned}$$

as desired.