2024 Paper 2

2024.2 Question 1

1. In the $n+k$ integers, the first one is c, and the final one is $c+n+k-1$.

   In the n integers, the first one is $c+n+k$, and the final one is $c+2n+k-1$.

   Hence, the sums are equal if and only if

$$\begin{aligned}\frac{(n+k)[c+(c+n+k-1)]}{2} &= \frac{n[(c+n+k)+(c+2n+k-1)]}{2} \\ (n+k)(2c+n+k-1) &= n(2c+3n+2k-1) \\ 2cn+n^2+nk-n+2ck+kn+k^2-k &= 2cn+3n^2+2kn-1 \\ 2ck+k^2 &= 2n^2+k,\end{aligned}$$

   as desired. All the above steps are reversible.

2. (a) When $k=1$, $2c+1=2n^2+1$, and $c=n^2$.
   Hence,
   $$(c,n) \in \left\{(t^2,t) \mid t \in \mathbb{N}\right\},$$
   and n can take all positive integers.

   (b) When $k=2$, $4c+4=2n^2+2$, and $2c=n^2-1$.
   By parity, n must be odd. Let $n=2t-1$ for $t \in \mathbb{N}$, and we have
   $$2c=(2t-1)^2-1=4t^2-4t,$$
   and hence
   $$c=2t^2-2t.$$
   Hence,
   $$(c,n) \in \left\{(2t^2-2t, 2t-1) \mid t \in \mathbb{N}\right\},$$
   and n can take all odd positive integers.

3. If $k=4$, we have $8c+16=2n^2+4$, and hence $n^2=4c+6$.

   By considering modulo 4, the only quadratic residues modulo 4 are 0 and 1, but the right-hand side equation is congruent to 2 modulo 4.

   Hence, there are no solutions for n and c.

4. When $c=1$, we have $2n^2+k=2k+k^2$, and hence $2n^2=k^2+k$.

   (a) When $k=1$, $k^2+k=2$, and so $(n,k)=(1,1)$ satisfies the equation.
   When $k=8$, $k^2+k=64+8=72$, and so $(n,k)=(6,8)$ satisfies the equation.

   (b) Given that $2N^2=K^2+K$, notice that
   $$\begin{aligned}(2N'^2)-(K'^2+K') &= 2(3N+2K+1)^2-(4N+3K+1)^2-(4N+3K+1) \\ &= 2(9N^2+4K^2+1+12NK+6N+4K) \\ &\quad -(16N^2+9K^2+1+24NK+8N+6K) \\ &\quad -(4N+3K+1) \\ &= 2N^2-K^2-K \\ &= 2N^2-(K^2+K) \\ &= 2N^2-2N^2 \\ &= 0,\end{aligned}$$
   and this means that
   $$2N'^2=K'^2+K',$$
   and hence
   $$(N',K')=(3N+2K+1, 4N+3K+1)$$
   is another pair of solution for (n,k).

   (c) When $(n,k)=(6,8)$, $3n+2k+1=35$, $4n+3k+1=49$, and $(n,k)=(35,49)$ is also possible.
   When $(n,k)=(35,49)$, $3n+2k+1=204, 4n+3k+1=288$, and $(n,k)=(204,288)$ is also possible.

2024.2 Question 2

1. By Newton's binomial theorem, we have

$$\begin{aligned}(8+x^3)^{-1} &= \frac{1}{8}\left(1+\left(\frac{x}{2}\right)^3\right)^{-1} \\ &= \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{3k},\end{aligned}$$

and this is valid for

$$\left|\frac{x}{2}\right| < 1, |x| < 2,$$

as desired.

Hence,

$$\begin{aligned}\int_0^1 \frac{x^m}{8+x^3}\,\mathrm{d}x &= \int_0^1 \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{3k} x^m\,\mathrm{d}x \\ &= \frac{1}{8}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3k}}\int_0^1 x^{3k+m}\,\mathrm{d}x \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left[\frac{x^{3k+m+1}}{3k+m+1}\right]_0^1 \\ &= \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+m+1}\right),\end{aligned}$$

as desired.

2. Let $m=2$, and we have

$$\int_0^1 \frac{x^2}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+3}\right).$$

Let $m=1$, and we have

$$\int_0^1 \frac{x}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+2}\right).$$

Let $m=0$, and we have

$$\int_0^1 \frac{x}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+1}\right).$$

Hence,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{1}{3k+3}-\frac{2}{3k+2}+\frac{4}{3k+1}\right) &= \int_0^1 \frac{x^2}{8+x^3}\,\mathrm{d}x - 2\int_0^1 \frac{x}{8+x^3}\,\mathrm{d}x + 4\int_0^1 \frac{\mathrm{d}x}{8+x^3} \\ &= \int_0^1 \frac{x^2-2x+4}{8+x^3}\,\mathrm{d}x \\ &= \int_0^1 \frac{x^2-2x+4}{(x+2)(x^2-2x+4)}\,\mathrm{d}x \\ &= \int_0^1 \frac{\mathrm{d}x}{x+2} \\ &= [\ln|x+2|]_0^1 \\ &= \ln 3 - \ln 2.\end{aligned}$$

3. Using partial fractions, let A' and B' be real constants such that

$$\begin{aligned}\frac{72(2k+1)}{(3k+1)(3k+2)} &= \frac{A'}{3k+1} + \frac{B'}{3k+2} \\ &= \frac{3(A'+B')k + (2A'+B')}{(3k+1)(3k+2)}.\end{aligned}$$

Hence, we have

$$\begin{cases} 3(A'+B') = 72 \cdot 2 = 144, \\ \quad 2A'+B' = 72. \end{cases}$$

Therefore, $(A', B') = (24, 24)$.

Let

$$A = \int_0^1 \frac{\mathrm{d}x}{8+x^3}, B = \int_0^1 \frac{x\,\mathrm{d}x}{8+x^3}, C = \int_0^1 \frac{x^2\,\mathrm{d}x}{8+x^3},$$

and what is desired is $24(A+B)$.

From the previous part, we can see that $4A - 2B + C = \ln 3 - \ln 2$.

Also,

$$\begin{aligned}2A + B &= \int_0^1 \frac{(2+x)\,\mathrm{d}x}{8+x^3} \\ &= \int_0^1 \frac{\mathrm{d}x}{x^2 - 2x + 4} \\ &= \int_0^1 \frac{\mathrm{d}x}{(x-1)^2 + 3} \\ &= \frac{1}{\sqrt{3}} \left[\arctan\left(\frac{x-1}{\sqrt{3}}\right)\right]_0^1 \\ &= \frac{1}{\sqrt{3}} \cdot \left[\arctan 0 - \arctan\left(-\frac{1}{\sqrt{3}}\right)\right] \\ &= \frac{1}{\sqrt{3}} \cdot \frac{\pi}{6} \\ &= \frac{\pi}{6\sqrt{3}}.\end{aligned}$$

We also have

$$\begin{aligned}C &= \int_0^1 \frac{x^2\,\mathrm{d}x}{8+x^3} \\ &= \frac{1}{3}\left[\ln(8+x^3)\right]_0^1 \\ &= \frac{1}{3}\left[\ln 9 - \ln 8\right] \\ &= \frac{2}{3}\ln 3 - \ln 2.\end{aligned}$$

Hence, we have

$$4A - 2B = \ln 3 - \ln 2 - \frac{2}{3}\ln 3 + \ln 2 = \frac{1}{3}\ln 3,$$

and hence $2A - B = \frac{1}{6}\ln 3$.

Therefore,

$$4A = \frac{1}{6}\ln 3 + \frac{\pi}{6\sqrt{3}},$$

and hence

$$A = \frac{\ln 3}{24} + \frac{\pi}{24\sqrt{3}}.$$

Subtracting two of this from $2A + B$ gives

$$B = \frac{\pi}{6\sqrt{3}} - \frac{\ln 3}{12} - \frac{\pi}{12\sqrt{3}} = \frac{\pi}{12\sqrt{3}} - \frac{\ln 3}{12},$$

and hence what is desired is

$$\begin{aligned}
24(A+B) &= 24\left(\frac{\pi}{24\sqrt{3}} + \frac{\pi}{12\sqrt{3}} + \frac{\ln 3}{24} - \frac{\ln 3}{12}\right) \\
&= 24\left(\frac{\pi}{8\sqrt{3}} - \frac{\ln 3}{24}\right) \\
&= \pi\sqrt{3} - \ln 3,
\end{aligned}$$

which gives $a = 3, b = 3$.

2024.2 Question 3

1. The line NP has gradient
$$m_{NP} = \frac{\sin\theta - 0}{\cos\theta - (-1)} = \frac{\sin\theta}{\cos\theta + 1},$$
and hence it has equation
$$l_{NP} : y = \frac{\sin\theta}{\cos\theta + 1} \cdot (x + 1).$$
When $x = 0$, we have
$$\begin{aligned} q &= \frac{\sin\theta}{\cos\theta + 1} \\ &= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} - 1 + 1} \\ &= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\ &= \tan\frac{\theta}{2}. \end{aligned}$$

2. (a)
$$\begin{aligned} \text{RHS} &= \tan\frac{1}{2}\left(\theta + \frac{1}{2}\pi\right) \\ &= \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \\ &= \frac{\tan\frac{\theta}{2} + \tan\frac{\pi}{4}}{1 - \tan\frac{\theta}{2}\tan\frac{\pi}{4}} \\ &= \frac{q+1}{1-q} \\ &= f_1(q), \end{aligned}$$
as desired.

   (b) Let the coordinates of P_1 be $(\cos\varphi, \sin\varphi)$, and hence we must have
$$\begin{aligned} f_1(q) &= \tan\frac{1}{2}\varphi \\ \tan\frac{1}{2}\left(\theta + \frac{1}{2}\pi\right) &= \tan\frac{1}{2}\varphi \\ \varphi &= \theta + \frac{1}{2}\pi, \end{aligned}$$
and so P_1 is the image of P being rotated through an angle of π counterclockwise about the origin.

3. (a) The coordinates of P_2 are $\left(\cos\left(\theta + \frac{1}{3}\pi\right), \sin\left(\theta + \frac{1}{3}\pi\right)\right)$, and hence we must have that
$$\begin{aligned} f_3(q) &= \tan\frac{1}{2}\left(\theta + \frac{1}{3}\pi\right) \\ &= \tan\left(\frac{\theta}{2} + \frac{\pi}{6}\right) \\ &= \frac{\tan\frac{\theta}{2} + \tan\frac{\pi}{6}}{1 - \tan\frac{\theta}{2}\tan\frac{\pi}{6}} \\ &= \frac{q + \frac{1}{\sqrt{3}}}{1 - q \cdot \frac{1}{\sqrt{3}}} \\ &= \frac{1 + \sqrt{3}q}{\sqrt{3} - q}. \end{aligned}$$

(b) Notice that $f_3(q) = f_1(-q) = \tan \frac{1}{2} \left(-\theta + \frac{1}{2}\pi\right)$, and so the coordinates of P_3 must be

$$\left(\cos\left(\frac{1}{2}\pi - \theta\right), \sin\left(\frac{1}{2}\pi - \theta\right)\right),$$

which is $P_3(\sin\theta, \cos\theta)$, a reflection of P in the line $y = x$.

(c) P_4 must be the image of P under the following transformations:

- Rotation counterclockwise by $\frac{1}{3}\pi$ about the origin O;
- Reflection in the line $y = x$;
- Rotation clockwise by $\frac{1}{3}\pi$ about the origin O.

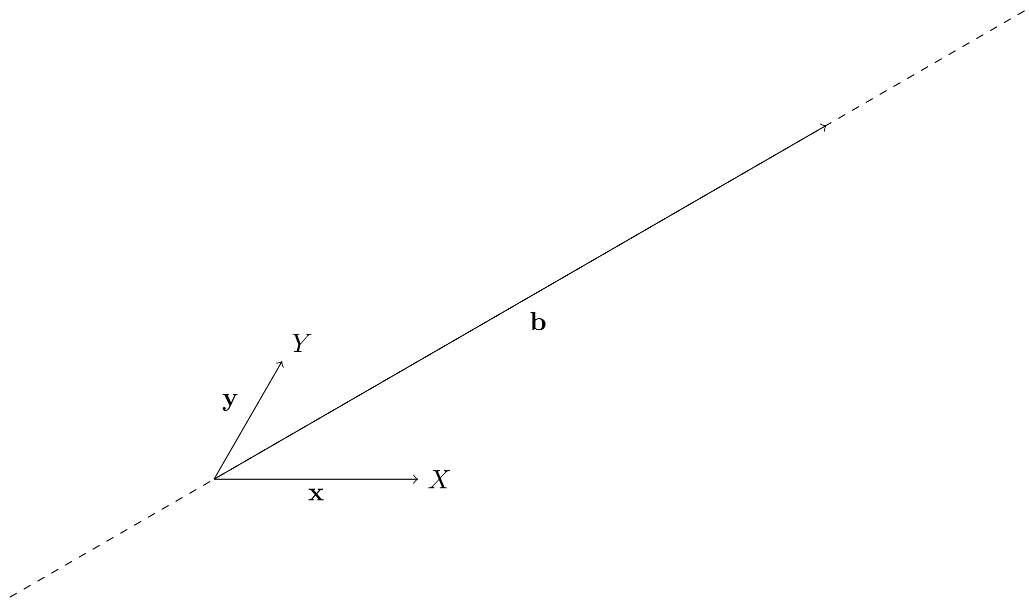
This is precisely the reflection in which the axis after the second step is $y = x$. Hence, the axis of this reflection has an angle of $\frac{1}{4}\pi - \frac{1}{3}\pi = \frac{1}{12}\pi$ with the positive x-axis.

P_4 is the image of P reflected in the line which makes an angle of $-\frac{\pi}{12}$ with the positive x-axis, passing through the origin.

2024.2 Question 4

1. (a) We first show that $\mathbf{b}$ lies in the plane XOY. Since $\mathbf{b}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$, it must lie in the plane containing $\mathbf{x} = \overrightarrow{OX}$ and $\mathbf{y} = \overrightarrow{OY}$, which is the plane XOY.
   Let α be the angle between $\mathbf{b}$ and $\mathbf{x}$, and let β be the angle between $\mathbf{b}$ and $\mathbf{y}$, where $0 \leq \alpha, \beta \leq \pi$.
   We have
   $$\begin{aligned}\cos\alpha &= \frac{\mathbf{b}\cdot\mathbf{x}}{|\mathbf{b}||\mathbf{x}|} \\ &= \frac{1}{|\mathbf{b}|}\cdot\frac{(|\mathbf{x}|\mathbf{y}+|\mathbf{y}|\mathbf{x})\cdot\mathbf{x}}{|\mathbf{x}|} \\ &= \frac{1}{|\mathbf{b}|}\cdot\frac{|\mathbf{x}|\cdot(\mathbf{x}\cdot\mathbf{y})+|\mathbf{y}|\cdot|\mathbf{x}|^2}{|\mathbf{x}|} \\ &= \frac{1}{|\mathbf{b}|}\cdot(\mathbf{x}\cdot\mathbf{y}+|\mathbf{x}|\cdot|\mathbf{y}|).\end{aligned}$$
   Similarly,
   $$\cos\beta = \frac{1}{|\mathbf{b}|}\cdot(\mathbf{x}\cdot\mathbf{y}+|\mathbf{x}|\cdot|\mathbf{y}|) = \cos\alpha.$$
   Since the cos function is one-to-one on $[0,\pi]$, we must have $\alpha = \beta$.
   Since $\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}|\cdot|\mathbf{y}|\cdot\cos\theta$ where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, we have $\mathbf{x}\cdot\mathbf{y} \geq -|\mathbf{x}||\mathbf{y}|$, and since $\theta \neq \pi$ (since OXY are non-collinear), we have $\mathbf{x}\cdot\mathbf{y} > -|\mathbf{x}||\mathbf{y}|$, and hence $\cos\alpha = \cos\beta > 0$.
   This shows that both angles are less than $\frac{\pi}{2} = 90°$.
   Hence, the three conditions
   - $\mathbf{b}$ lies in the plane OXY,
   - the angle between $\mathbf{b}$ and $\mathbf{x}$ is equal to the angle between $\mathbf{b}$ and $\mathbf{y}$,
   - both angles are less than $\frac{\pi}{2} = 90°$

   are all satisfied, and we can conclude that $\mathbf{b}$ is a bisecting vector for the plane OXY.



   All bisecting vectors must lie on the line containing $\mathbf{b}$ (the dashed line on the diagram), and hence a scalar multiple of $\mathbf{b}$.
   Furthermore, since both angles must be less than $\frac{\pi}{2}$, it must not on the opposite as where $\mathbf{b}$ is situated, and hence it must be a positive multiple of $\mathbf{b}$.

   (b) If B lies on XY, then $\mathbf{OB} = \mu\mathbf{x} + (1-\mu)\mathbf{y}$ must be a convex combination of $\mathbf{x}$ and $\mathbf{y}$, and hence
   $$\lambda(|\mathbf{x}|\mathbf{y}+|\mathbf{y}|\mathbf{x}) = \mu\mathbf{x}+(1-\mu)\mathbf{y}.$$

Since O, X and Y are not collinear, we must have $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and hence $\lambda|\mathbf{y}| = \mu$ and $\lambda|\mathbf{x}| = 1 - \mu$, hence giving

$$\lambda = \frac{1}{|\mathbf{x}| + |\mathbf{y}|}$$

We therefore have

$$\begin{aligned}
\frac{XB}{BY} &= \frac{\left|\overrightarrow{OB} - \mathbf{x}\right|}{\left|\mathbf{y} - \overrightarrow{OB}\right|} \\
&= \frac{\left|\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y} + \frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y}\mathbf{x} - \mathbf{x}\right|}{\left|\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y} + \frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y}\mathbf{x} - \mathbf{y}\right|} \\
&= \frac{\left|\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|}(\mathbf{y} - \mathbf{x})\right|}{\left|\frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|}(\mathbf{x} - \mathbf{y})\right|} \\
&= \frac{\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|} \cdot |\mathbf{y} - \mathbf{x}|}{\frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|} \cdot |\mathbf{x} - \mathbf{y}|} \\
&= \frac{|\mathbf{x}|}{|\mathbf{y}|},
\end{aligned}$$

which means

$$XB : BY = |\mathbf{x}| : |\mathbf{y}|,$$

which is precisely the angle bisector theorem.

(c) Considering the dot product,

$$\begin{aligned}
\overrightarrow{OB} \cdot \overrightarrow{XY} &= \lambda\mathbf{b} \cdot (\mathbf{y} - \mathbf{x}) \\
&= \lambda\left(|\mathbf{x}|\mathbf{y} + |\mathbf{y}|\mathbf{x}\right) \cdot (\mathbf{y} - \mathbf{x}) \\
&= \lambda\left[|\mathbf{x}| \cdot \mathbf{y} \cdot \mathbf{y} + |\mathbf{y}| \cdot \mathbf{x} \cdot \mathbf{y} - |\mathbf{x}| \cdot \mathbf{x} \cdot \mathbf{y} - |\mathbf{y}| \cdot \mathbf{x} \cdot \mathbf{x}\right] \\
&= \lambda\left[|\mathbf{x}| \cdot |\mathbf{y}|^2 + \left[|\mathbf{y}| - |\mathbf{x}|\right]\mathbf{x} \cdot \mathbf{y} - |\mathbf{y}| \cdot |\mathbf{x}|^2\right] \\
&= \lambda\left(|\mathbf{y}| - |\mathbf{x}|\right)\left(|\mathbf{x}||\mathbf{y}| + \mathbf{x} \cdot \mathbf{y}\right) \\
&= 0.
\end{aligned}$$

Since O, X, Y are not collinear, $\mathbf{x} \cdot \mathbf{y} > -|\mathbf{x}||\mathbf{y}|$, and hence $|\mathbf{x}||\mathbf{y}| + \mathbf{x} \cdot \mathbf{y} > 0$.
Also, $\lambda = \frac{1}{|\mathbf{x}|+|\mathbf{y}|} \neq 0$.
So it must be the case that $|\mathbf{x}| - |\mathbf{y}| = 0$, which means $|\mathbf{x}| = |\mathbf{y}|$.
Hence, $OX = OY$, and triangle OXY is isosceles.

2. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the bisecting vectors for QOR, ROP and POQ respectively, and let $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{q} = \overrightarrow{OQ}$, $\mathbf{r} = \overrightarrow{OR}$.

   Let i, j, k be some arbitrary positive real constant.

   From the question, we have

$$\begin{cases}
\mathbf{u} = i\left(|\mathbf{q}|\mathbf{r} + |\mathbf{r}|\mathbf{q}\right), \\
\mathbf{v} = j\left(|\mathbf{r}|\mathbf{p} + |\mathbf{p}|\mathbf{r}\right), \\
\mathbf{w} = k\left(|\mathbf{p}|\mathbf{q} + |\mathbf{q}|\mathbf{p}\right).
\end{cases}$$

   Considering a pair of dot-product, we have

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= ij \cdot \left(|\mathbf{q}||\mathbf{r}|\mathbf{r} \cdot \mathbf{p} + |\mathbf{p}||\mathbf{q}|\mathbf{r} \cdot \mathbf{r} + |\mathbf{r}||\mathbf{r}|\mathbf{p} \cdot \mathbf{q} + |\mathbf{r}||\mathbf{p}|\mathbf{q} \cdot \mathbf{r}\right) \\
&= ij|\mathbf{r}|\left(|\mathbf{q}|\mathbf{r} \cdot \mathbf{q} + |\mathbf{p}|\mathbf{r} \cdot \mathbf{q} + |\mathbf{p}||\mathbf{q}||\mathbf{r}| + |\mathbf{r}|\mathbf{p} \cdot \mathbf{q}\right) \\
&= ij|\mathbf{r}|^2|\mathbf{p}||\mathbf{q}|\left(\cos\langle\mathbf{p}, \mathbf{r}\rangle + \cos\langle\mathbf{r}, \mathbf{q}\rangle + \cos\langle\mathbf{p}, \mathbf{q}\rangle + 1\right),
\end{aligned}$$

where $\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the angle between $\mathbf{a}$ and $\mathbf{b}$, in $[0, \pi]$.

Denote

$$t = \cos\langle \mathbf{p}, \mathbf{r} \rangle + \cos\langle \mathbf{r}, \mathbf{q} \rangle + \cos\langle \mathbf{q}, \mathbf{p} \rangle + 1,$$

and hence

$$\begin{cases} \mathbf{u} \cdot \mathbf{v} = ij|\mathbf{r}|^2|\mathbf{p}||\mathbf{q}|t, \\ \mathbf{u} \cdot \mathbf{w} = ik|\mathbf{r}||\mathbf{p}||\mathbf{q}|^2 t, \\ \mathbf{v} \cdot \mathbf{w} = jk|\mathbf{r}||\mathbf{p}|^2|\mathbf{q}|t. \end{cases}$$

Since $i, j, k > 0$, and $|\mathbf{p}|, |\mathbf{q}|, |\mathbf{r}| > 0$ since none of P, Q, R are at O, we must have

$$\operatorname{sgn}(\mathbf{u} \cdot \mathbf{v}) = \operatorname{sgn}(\mathbf{u} \cdot \mathbf{w}) = \operatorname{sgn}(\mathbf{v} \cdot \mathbf{w}) = \operatorname{sgn} t,$$

where $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, -1\}$ is the sign function defined as

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

But the sign of a dot product also corresponds to the angle between two non-collinear non-zero vectors, since this resembles the sign of the cosine of the angle between them:

$$\begin{aligned} \operatorname{sgn} \mathbf{a} \cdot \mathbf{b} &= \operatorname{sgn} |\mathbf{a}||\mathbf{b}| \cos\langle \mathbf{a}, \mathbf{b} \rangle \\ &= \operatorname{sgn} \cos\langle \mathbf{a}, \mathbf{b} \rangle \\ &= \begin{cases} 1, & \langle a, b \rangle \text{ is acute}, \\ 0, & \langle a, b \rangle \text{ is right-angle}, \\ -1, & \langle a, b \rangle \text{ is obtuse}. \end{cases} \end{aligned}$$

This means the angles between $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{v}$ and $\mathbf{w}$ must all be acute, obtuse, or right angles. This is exactly what is desired, and finishes our proof.

2024.2 Question 5

1. We have
$$f_1(n) = n^2 + 6n + 11 = (n+3)^2,$$
and so
$$f_1(\mathbb{Z}) = \{(n+3)^2 + 2 \mid n \in \mathbb{Z}\}.$$
But since if $n \in \mathbb{Z}, n+3 \in \mathbb{Z}$, and if $n+3 \in \mathbb{Z}, n \in \mathbb{Z}$, so
$$f_1(\mathbb{Z}) = \{(n+3)^2 + 2 \mid n \in \mathbb{Z}\} = \{n^2 + 2 \mid n \in \mathbb{Z}\}.$$
We have $F_1(\mathbb{Z}) = \{n^2 + 2 \mid n \in \mathbb{Z}\}$, and so $f_1(\mathbb{Z}) = F_1(\mathbb{Z})$, which shows f_1 and F_1 has the same range/

2. We have
$$g_1(n) = n^2 - 2n + 5 = (n-1)^2 + 4,$$
and so
$$g_1(\mathbb{Z}) = \{(n-1)^2 + 4 \mid n \in \mathbb{Z}\} = \{n^2 + 4 \mid n \in \mathbb{Z}\}.$$
The quadratic residues modulo 4 are 0 and 1, and so
$$f_1(\mathbb{Z}) \subseteq \{0+2, 1+2\} = \{2, 3\} \bmod 4,$$
and
$$g_1(\mathbb{Z}) \subseteq \{0+4, 1+4\} = \{0, 1\} \bmod 4.$$
Under modulo 4, $f_1(\mathbb{Z}) \cap g_1(\mathbb{Z}) \subseteq \{2, 3\} \cap \{0, 1\} = \varnothing$.
Hence, $f_1(\mathbb{Z}) \cap g_1(\mathbb{Z}) = \varnothing$ under modulo 4, and hence $f_1(\mathbb{Z}) \cap g_1(\mathbb{Z}) = \varnothing$.

3. We have
$$f_2(n) = n^2 - 2n - 6 = (n-1)^2 - 7,$$
and so
$$f_2(\mathbb{Z}) = \{(n-1)^2 - 7 \mid n \in \mathbb{Z}\} = \{n^2 - 7 \mid n \in \mathbb{Z}\}.$$
Similarly,
$$g_2(n) = n^2 - 4n + 2 = (n-2)^2 - 2,$$
and so
$$g_2(\mathbb{Z}) = \{(n-2)^2 - 2 \mid n \in \mathbb{Z}\} = \{n^2 - 2 \mid n \in \mathbb{Z}\}.$$
So for the intersection, if $t \in f_2(\mathbb{Z}) \cap g_2(\mathbb{Z})$, then there exists $n_1, n_2 \in \mathbb{Z}$,
$$t = n_1^2 - 7 = n_2^2 - 2,$$
and hence
$$n_1^2 - n_2^2 = (n_1 + n_2)(n_1 - n_2) = 5.$$
So
$$(n_1 + n_2, n_1 - n_2) = (\pm 1, \pm 5) \text{ or } (\pm 5, \pm 1),$$
and hence
$$(n_1, n_2) = (\pm 3, \mp 2) \text{ or } (\pm 3, \pm 2),$$
which gives
$$t = (\pm 3)^2 - 7 = 2.$$
Therefore,
$$f_2(\mathbb{Z}) \cap g_2(\mathbb{Z}) = \{2\},$$
and 2 is the only integer which lies in the intersection of the range of f_2 and g_2.

4. Since $p, q \in \mathbb{R}$, we must have $p+q, p-q \in \mathbb{R}$ and hence

$$\begin{aligned}(p+q)^2 &= p^2 + 2pq + q^2 \geq 0,\\ (p-q)^2 &= p^2 - 2pq + q^2 \geq 0.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{3}{4}(p+q)^2 + \frac{1}{4}(p-q)^2 &= \frac{3}{4}\left(p^2+2pq+q^2\right) + \frac{1}{4}\left(p^2-2pq+q^2\right)\\ &= p^2+pq+q^2\\ &\geq 0,\end{aligned}$$

as desired.

We have

$$f_3(n) = n^3 - 3n^2 + 7n = (n-1)^3 + 4n + 1 = (n-1)^3 + 4(n-1) + 5,$$

and so

$$f_3(\mathbb{Z}) = \{(n-1)^3 + 4(n-1) + 5 \mid n \in \mathbb{Z}\} = \{n^3 + 4n + 5 \mid n \in \mathbb{Z}\}.$$

We have

$$g_3(\mathbb{Z}) = \{n^3 + 4n - 6 \mid n \in \mathbb{Z}\}.$$

So if $t \in f_3(\mathbb{Z}) \cap g_3(\mathbb{Z})$, then there exists $n_1, n_2 \in \mathbb{Z}$ such that

$$t = n_1^3 + 4n_1 + 5 = n_2^3 + 4n_2 - 6.$$

Hence,

$$(n_1^3 - n_2^3) + 4(n_1 - n_2) = (n_1 - n_2)(n_1^2 + n_1n_2 + n_2^2 + 4) = -11.$$

Since $n_1^2 + n_1n_2 + n_2^2 \geq 0$ by the lemma in the previous part, we have $n_1^2 + n_1n_2 + n_2^2 + 4 \geq 4$. But $n_1^2 + n_1n_2 + n_2^2 + 4 \mid -11$, and so

$$n_1^2 + n_1n_2 + n_2^2 + 4 = 11, n_1 - n_2 = -1.$$

Putting $n_2 = n_1 + 1$ into the first equation, we have

$$\begin{aligned}n_1^2 + n_1n_2 + n_2^2 + 4 &= n_1^2 + n_1(n_1+1) + (n_1+1)^2 + 4\\ &= n_1^2 + n_1^2 + n_1 + n_1^2 + 2n_1 + 1 + 4\\ &= 3n_1^2 + 3n_1 + 5\\ &= 11,\end{aligned}$$

and hence

$$3n_1^2 + 3n_1 - 6 = 3(n_1+2)(n_1-1) = 0,$$

which gives $n_1 = -2$ or $n_1 = 1$, and they correspond to $n_2 = -1$ or $n_2 = 2$.

Hence,

$$t = (-1)^3 + 4(-1) - 6 = -1 - 4 - 6 = -11,$$

or

$$t = 2^3 + 4 \cdot 2 - 6 = 8 + 8 - 6 = 10.$$

Hence,

$$f_3(\mathbb{Z}) \cap g_3(\mathbb{Z}) = \{-11, 10\},$$

and the integers that lie in the intersection of the ranges are -11 and 10.

2024.2 Question 6

1. We first look at the base case where $n = 0$, and we have

$$\text{RHS} = \frac{1}{2^{2\cdot 0}}\binom{2\cdot 0}{0} = \frac{1}{2^0}\binom{0}{0} = 1,$$

and LHS $= T_0 = 1$. So the desired statement is satisfied for the base case where $n = 0$.

Assume the original statement is true for some $n = k \geq 0$, that

$$T_n = \frac{1}{2^{2n}}\binom{2n}{n}.$$

Consider $n = k + 1$, we have

$$\begin{aligned} T_n &= T_{k+1} \\ &= \frac{2(k+1)-1}{2(k+1)}T_k \\ &= \frac{2k+1}{2(k+1)} \cdot \frac{1}{2^{2k}}\binom{2k}{k} \\ &= \frac{(2k+1)(2k+2)}{2(k+1)2(k+1)} \cdot \frac{1}{2^{2k}}\frac{(2k)!}{(}k!k!) \\ &= \frac{(2k+2)!}{(k+1)!(k+1)!} \cdot \frac{1}{2^{2k+2}} \\ &= \frac{1}{2^{2(k+1)}}\binom{2(k+1)}{k+1}, \end{aligned}$$

which is precisely the statement for $n = k + 1$.

The original statement is true for $n = 0$, and given it holds for some $n = k \geq 0$, it holds for $n = k + 1$. Hence, by the principle of mathematical induction, the statement

$$T_n = \frac{1}{2^{2n}}\binom{2n}{n}$$

holds for all integers $n \geq 0$, as desired.

2. By Newton's binomial theorem, we have

$$(1-x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x)^3 + \cdots,$$

and notice that the negative signs cancels out, and hence

$$a_n = \frac{\prod_{k=1}^{n}\frac{2k-1}{2}}{n!} = \frac{\prod_{k=1}^{n}(2k-1)}{2^n n!}.$$

Hence, we note that

$$\begin{aligned} \frac{a_r}{a_{r-1}} &= \frac{\prod_{k=1}^{r}(2k-1)/(2^r r!)}{\prod_{k=1}^{r-1}(2k-1)/(2^{r-1}(r-1)!)} \\ &= \frac{2r-1}{2r}, \end{aligned}$$

and hence

$$a_r = \frac{2r-1}{2r}a_{r-1}.$$

Note that $a_0 = 1$ as well. The sequence $\{a_n\}_0^\infty$ and $\{T_n\}_0^\infty$ have the same initial term $a_0 = T_0 = 1$, and they have the same same inductive relationship

$$a_n = \frac{2n-1}{2n}a_{n-1}, T_n = \frac{2n-1}{2n}T_{n-1}.$$

This shows they are the same sequence, hence

$$a_n = T_n$$

for all $n = 0, 1, 2, \cdots$.

3. By Newton'w binomial theorem,

$$(1-x)^{-\frac{3}{2}} = 1 + \frac{\left(-\frac{3}{2}\right)(-x)}{1!} + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)}{2!} + \cdots,$$

and so

$$b_n = \frac{\prod_{k=1}^{n} \frac{2k+1}{2}}{n!} = \frac{\prod_{k=1}^{n}(2k+1)}{2^n n!}.$$

Notice that

$$\begin{aligned}\frac{b_n}{a_n} &= \frac{\prod_{k=1}^{n}(2k+1)/(2^n n!)}{\prod_{k=1}^{n}(2k-1)/2^n n!} \\ &= \frac{\prod_{k=1}^{n}(2k+1)}{\prod_{k=1}^{n}(2k-1)} \\ &= \frac{\prod_{k=2}^{n+1}(2k-1)}{\prod_{k=1}^{n}(2k-1)} \\ &= \frac{2(n+1)-1}{2 \cdot 1 - 1} \\ &= 2n+1,\end{aligned}$$

and so

$$\begin{aligned}b_n &= (2n+1)a_n \\ &= (2n+1) \cdot \frac{1}{2^{2n}} \cdot \binom{2n}{n} \\ &= \frac{2n+1}{2^{2n}} \binom{2n}{n}.\end{aligned}$$

4. By the binomial expansion, we have

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots,$$

and we have

$$(1-x)^{-\frac{1}{2}} \cdot (1-x)^{-1} = (1-x)^{-\frac{3}{2}}.$$

For a particular term in the series expansion for $(1-x)^{-\frac{3}{2}}$, say b_n, we must have

$$b_n x^n = \sum_{t=0}^{n} a_t \cdot x^t \cdot 1 \cdot x^{n-t},$$

and hence

$$b_n = \sum_{t=0}^{n} a_t,$$

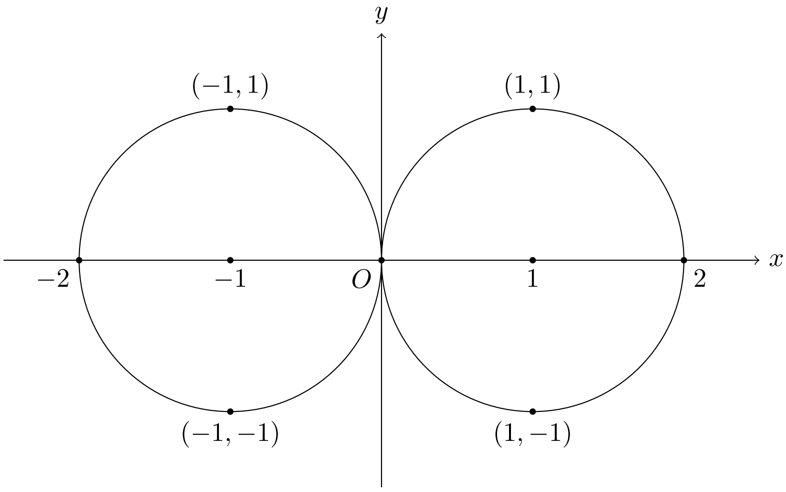
which gives

$$\frac{2n+1}{2^{2n}} \binom{2n}{n} = \sum_{r=0}^{n} \frac{1}{2^{2r}} \binom{2r}{r},$$

exactly as desired.

2024.2 Question 7

1. In this case, we have either $y^2 + (x-1)^2 = 1$ (giving a circle with radius 1 centred at $(1, 0)$), or $y^2 + (x+1)^2 = 1$ (giving a circle with radius 1 centred at $(-1, 0)$).



2. At $y = k$, we have

$$
\begin{aligned}
[(x-1)^2 + (k^2-1)][(x+1)^2 + (k^2-1)] &= \frac{1}{16} \\
(x-1)^2(x+1)^2 + (k^2-1)[(x-1)^2 + (x+1)^2] + (k^2-1)^2 - \frac{1}{16} &= 0 \\
(x^2-1)^2 + 2(k^2-1)(x^2+1) + (k^4 - 2k^2 + 1) - \frac{1}{16} &= 0 \\
x^4 - 2x^2 + 1 + 2(k^2-1)x^2 + 2(k^2-1) + (k^4 - 2k^2 + 1) - \frac{1}{16} &= 0 \\
x^4 + 2(k^2-2)x^2 + k^4 - \frac{1}{16} &= 0,
\end{aligned}
$$

as desired.

By completing the square, we have

$$
\begin{aligned}
(x^2 + (k^2-2))^2 + k^4 - \frac{1}{16} - (k^2-2)^2 &= 0 \\
(x^2 + (k^2-2))^2 &= \frac{65}{16} - 4k^2.
\end{aligned}
$$

- If $4k^2 > \frac{65}{16}$, i.e. $k^2 > \frac{65}{64}$, the right-hand side is negative, so there will be no intersections.
- If $4k^2 = \frac{65}{16}$, i.e. $k^2 = \frac{65}{64}$, we have

$$x^2 + (k^2-2) = 0,$$

and hence

$$x^2 = 2 - k^2 = 2 - \frac{65}{64} = \frac{63}{64},$$

giving

$$x = \pm\frac{3\sqrt{7}}{8}.$$

There will be two intersections.
- If $4k^2 < \frac{65}{16}$, i.e. $k^2 < \frac{65}{64}$, we have

$$x^2 + (k^2-2) = \pm\sqrt{\frac{65}{16} - 4k^2},$$

and hence

$$x^2 = 2 - k^2 \pm \sqrt{\frac{65}{16} - 4k^2}.$$

The case where

$$\begin{aligned} x^2 &= 2 - k^2 + \sqrt{\frac{65}{16} - 4k^2} \\ &> 2 - k^2 \\ &> 2 - \frac{65}{64} \\ &= \frac{63}{64} \\ &> 0 \end{aligned}$$

always gives two solutions for x.

- If $2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} < 0$,

$$\begin{aligned} 2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} &< 0 \\ \sqrt{\frac{65}{16} - 4k^2} &> 2 - k^2 \\ \frac{65}{16} - 4k^2 &> k^4 - 4k^2 + 4 \\ k^4 &< \frac{1}{16} \\ k^2 &< \frac{1}{4}, \end{aligned}$$

  there are no solutions for the case where the minus sign is taken.
- If $2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} = 0$, $k^2 = \frac{1}{4}$, the minus sign produce precisely one solution $x = 0$, giving 3 intersections in total.
- If $2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} < 0$, $k^2 > \frac{1}{4}$, the minus sign will produce two additional roots, hence giving 4 intersections in total.

To summarise, the number of intersections with the line $y = k$ for each positive value of k is

$$\text{number of intersections} = \begin{cases} 0, & k^2 > \frac{65}{64}, k > \frac{\sqrt{65}}{8}, \\ 2, & k^2 = \frac{65}{64}, k = \frac{\sqrt{65}}{8}, \\ 4, & \frac{1}{4} < k^2 < \frac{65}{64}, \frac{1}{2} < k < \frac{\sqrt{65}}{8}, \\ 3, & k^2 = \frac{1}{4}, k = \frac{1}{2}, \\ 2, & k^2 < \frac{1}{4}, 0 < k < \frac{1}{2}. \end{cases}$$

3. When the point on C_2 has the greatest possible y-coordinate, the two points have x-coordinates

$$x = \pm\frac{3\sqrt{7}}{8},$$

and on C_1 has

$$x = \pm 1.$$

Since $3\sqrt{7} = \sqrt{63} < \sqrt{64} = 8$, we must have $\frac{3\sqrt{7}}{8} < 1$, meaning those on C_2 are closer to the y-axis than those on C_1.

4. If both are negative, then the distance from (x, y) to $(1, 0)$ and $(-1, 0)$ are both less than 1. But this is not possible, since the distance from $(1, 0)$ to $(-1, 0)$ is 2, which means the sum of the distances from (x, y) to those points has to be at least 2.

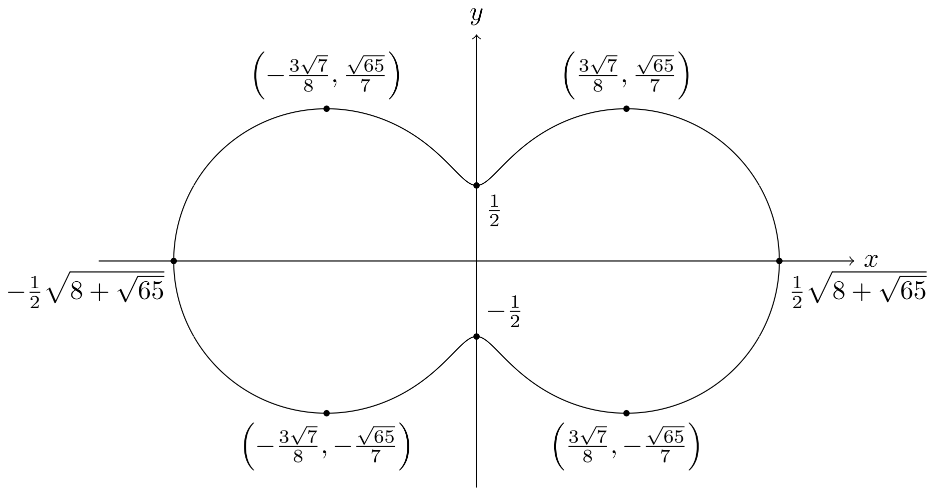
Therefore, since the product of those two terms are positive for C_2, and they cannot be both negative, they must both be positive, and hence the distance from (x, y) to $(1, 0)$ and $(-1, 0)$ are both more than 1, meaning all points on C_2 lies outside the two circles that make up C_1, which shows that C_2 lies entirely outside C_1.

5. C_2 is symmetric about both the x-axis and the y-axis.

   When $x = 0$, $y^4 = \frac{1}{16}$, and hence $y = \pm\frac{1}{2}$.

   When $y = 0$, $x^2 = 2 + \frac{\sqrt{65}}{16}$, and hence $x = \pm\sqrt{2 + \frac{\sqrt{65}}{4}} = \pm\frac{1}{2}\sqrt{8 + \sqrt{65}}$.

   Hence, the graph looks as follows.



2024.2 Question 8

1. Notice that by expanding this square,

$$\begin{aligned}(\sqrt{x_n}-\sqrt{y_n})^2 &= x_n+y_n-2\sqrt{x_ny_n}\\ &= 2a(x_n,y_n)-2g(x_n,y_n)\\ &= 2(x_{n+1}-y_{n+1}).\end{aligned}$$

Since this is a square, it must be non-negative, with the equal sign taking if and only if $\sqrt{x_n}=\sqrt{y_n}$, which holds if and only if $x_n=y_n$.

So $x_{n+1}\geq y_{n+1}$, and $x_{n+1}=y_{n+1}$ if and only if $x_n=y_n$.

Since $y_0<x_0$, we have $y_0\neq x_0$, and hence $y_1\neq x_1$. By induction, this shows that $y_n\neq_n$ for all n, and hence for all $n\geq 0$, $y_n<x_n$.

Furthermore,

$$\begin{aligned}x_n-x_{n+1} &= x_n-a(x_n,y_n)\\ &= x_n-\frac{x_n+y_n}{2}\\ &= \frac{x_n-y_n}{2}\\ &> 0,\end{aligned}$$

since $x_n>y_n$ and hence $x_n>x_{n+1}$.

Similarly,

$$\begin{aligned}y_{n+1}-y_n &= g(x_n,y_n)-y_N\\ &= \sqrt{x_ny_n}-y_N\\ &= \sqrt{y_n}(\sqrt{x_n}-\sqrt{y_n})\\ &> 0,\end{aligned}$$

since $x_n>y_n$ implies $\sqrt{x_n}>\sqrt{y_n}$, and hence $y_n<y_{n+1}$.

Hence, for all $n\in\mathbb{N}$,

$$y_n<x_n<x_{n-1}<x_{n-2}<\cdots<x_0,$$

and $y_{n+1}>y_n$.

Hence, $\{y_N\}_{n=0}^{\infty}$ is an increasing sequence, and is bounded above by x_0.

So there exists $M\in\mathbb{R}$ such that

$$\lim_{n\to\infty}y_n=M.$$

As for the inequality, the left inequality sign is equivalent to $y_{n+1}<x_{n+1}$ which was shown above.

To show the right inequality sign, this is equivalent to showing

$$\begin{aligned}\frac{1}{2}(\sqrt{x_n}-\sqrt{y_n})^2 &< \frac{1}{2}(x_n-y_n)\\ x_n+y_n-2\sqrt{x_ny_N} &< x_n-y_n\\ 2y_n &< 2\sqrt{x_ny_n}\\ \sqrt{y_n} &< \sqrt{x_n},\end{aligned}$$

which is true since $y_n<x_n$.

Hence,

$$0<x_{n+1}-y_{n+1}<\frac{1}{2}(x_n-y_n)$$

as desired.

Hence, we have

$$\begin{aligned}0 &< x_n - y_n \\ &< \frac{1}{2}(x_{n-1} - y_{n-1}) \\ &< \frac{1}{4}(x_{n-2} - y_{n-2}) \\ &< \cdots \\ &< \frac{1}{2^n}(x_0 - y_0),\end{aligned}$$

by induction.

$x_0 - y_0 > 0$ is a positive real constant. Let $n \to \infty$, and by the squeeze theorem, the strict inequalities become weak, and

$$0 \leq \lim_{n\to\infty} (x_n - y_n) \leq \lim_{n\to\infty} \left(\frac{1}{2^n}(x_0 - y_0)\right) = 0,$$

and hence

$$\lim_{n\to\infty} (x_n - y_n) = 0.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty} [(x_n - y_n) + y_n] \\ &= \lim_{n\to\infty} (x_n - y_n) + \lim_{n\to\infty} y_n \\ &= 0 + M \\ &= M,\end{aligned}$$

since both parts of the limit $x_n - y_n$ and y_n exist, the limit of the sum is the sum of the limits of the individual parts.

2. Using this substitution, when $x \to 0^+$, we have $t \to -\infty$, and when $x \to +\infty$, $t \to +\infty$. Also,

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2} + \frac{1}{2} \cdot \frac{pq}{x^2} = \frac{1}{2}\left(1 + \frac{pq}{x^2}\right).$$

Hence, the integral can be simplified as

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{\left(\frac{1}{4}(p+q)^2 + t^2\right)(pq + t^2)}} \\ &= \int_0^{\infty} \frac{\frac{1}{2}\left(1 + \frac{pq}{x^2}\right)\mathrm{d}x}{\sqrt{\left(\frac{1}{4}(p+q)^2 + \frac{1}{4}\left(x - \frac{pq}{x}\right)^2\right)\left(pq + \frac{1}{4}\left(x - \frac{pq}{x}\right)^2\right)}} \\ &= \int_0^{\infty} \frac{\frac{1}{2}\left(1 + \frac{pq}{x^2}\right)\mathrm{d}x}{\frac{1}{4}\sqrt{\left(p^2 + 2pq + q^2 + x^2 - 2pq + \frac{p^2q^2}{x^2}\right)\left(4pq + x^2 - 2pq + \frac{p^2q^2}{x^2}\right)}} \\ &= 2\int_0^{\infty} \frac{\left(1 + \frac{pq}{x^2}\right)\mathrm{d}x}{\sqrt{\left(p^2 + q^2 + x^2 + \frac{p^2q^2}{x^2}\right)\left(x^2 + 2pq + \frac{p^2q^2}{x^2}\right)}} \\ &= 2\int_0^{\infty} \frac{(x^2 + pq)\,\mathrm{d}x}{\sqrt{(x^4 + (p^2+q^2)x^2 + p^2q^2)(x^4 + 2pqx^2 + p^2q^2)}} \\ &= 2\int_0^{\infty} \frac{(x^2 + pq)\,\mathrm{d}x}{\sqrt{(x^2+p^2)(x^2+q^2)(x^2+pq)^2}} \\ &= 2\int_0^{\infty} \frac{\mathrm{d}x}{\sqrt{(x^2+p^2)(x^2+q^2)}} \\ &= 2I(p,q),\end{aligned}$$

which means
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{\left(\frac{1}{4}(p+q)^2+t^2\right)(pq+t^2)}} = 2I(p,q).$$

But also note that the left-hand side satisfies that
$$\begin{aligned}
\text{LHS} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{\left(\frac{1}{4}(p+q)^2+t^2\right)(pq+t^2)}} \\
&= 2\int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{\left[\left(\frac{1}{2}(p+q)\right)^2+t^2\right]\left[(\sqrt{pq})^2+t^2\right]}} \\
&= 2\int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{[a(p,q)^2+t^2]\,[g(p,q)^2+t^2]}} \\
&= 2I(a(p,q),g(p,q)),
\end{aligned}$$

since the integrand is an even function, and so
$$I(p,q) = I(a(p,q),g(p,q)),$$

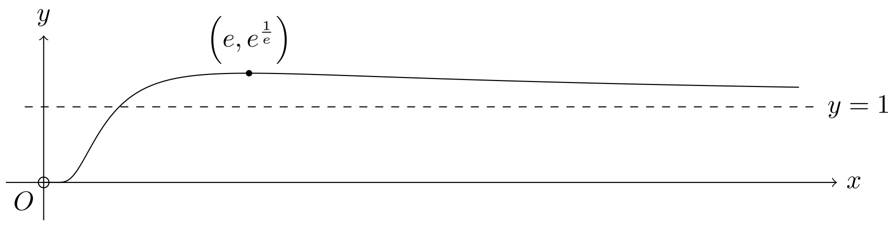
as desired.

Since $0<q<p$, let $y_0=q, x_0=p$, and hence
$$\begin{aligned}
I(p,q) &= I(x_0,y_0) \\
&= I(a(x_0,y_0),g(x_0,y_0)) \\
&= I(x_1,y_1) \\
&= \cdots \\
&= I(x_n,y_n).
\end{aligned}$$

Let $n\to\infty$, and we have
$$\begin{aligned}
I(p,q) &= I(M,M) \\
&= \int_0^{\infty} \frac{\mathrm{d}x}{M^2+x^2} \\
&= \frac{1}{M}\left[\arctan\left(\frac{x}{M}\right)\right]_0^{\infty} \\
&= \frac{\pi}{2M}.
\end{aligned}$$

2024.2 Question 11

1. Notice that
$$x^{\frac{1}{x}} = \exp\left(\frac{\ln x}{x}\right).$$
As $x \to 0^+$, $\frac{\ln x}{x} \to -\infty$, and hence $x^{\frac{1}{x}} \to 0^+$.
As $x \to \infty$, $\frac{\ln x}{x} \to 0^+$, and hence $x^{\frac{1}{x}} \to 1$.
We have
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\exp\left(\frac{\ln x}{x}\right)\\ &= x^{\frac{1}{x}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln x}{x}\\ &= x^{\frac{1}{x}} \cdot \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2}\\ &= x^{\frac{1}{x}} \cdot \frac{1-\ln x}{x^2}.\end{aligned}$$
This shows that $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ when $x > e$, $= 0$ when $x = e$, and > 0 when $x < e$.
This means that the point $\left(e, e^{\frac{1}{e}}\right)$ is a maximum for the graph.
Hence, the graph looks as follows.



The maximum of $n^{\frac{1}{n}}$ must occur for $n \in \mathbb{N}$ when $n = 2$ or $n = 3$, since $2 < e < 3$.
Notice that
$$\begin{aligned}2^{\frac{1}{2}} < 3^{\frac{1}{3}} &\iff 2^3 < 3^2\\ &\iff 8 < 9,\end{aligned}$$
which is true, so the maximum of $n^{\frac{1}{n}}$ occurs when $n = 3$.

2. Let X_i be the number of tests for each group, and let X be the total number of tests, we have
$$X = \sum_{i=1}^{r} X_i.$$
For each X_i, we have if the enzyme is not present in any of the persons, then there is only one test needed. Otherwise, if the enzyme is present in any of the persons, then an additional k tests are needed. Hence,
$$\mathrm{E}(X_i) = (1-p)^k + (1-(1-p)^k)(1+k) = 1 + (1-p^k)k,$$

and the expected total number of tests is given as

$$\begin{aligned}
\mathrm{E}(X) &= \mathrm{E}\left(\sum_{i=1}^{r} X_i\right) \\
&= \sum_{i=1}^{r} \mathrm{E}\left(X_i\right) \\
&= \sum_{i=1}^{r} \left[1 + (1-(1-p)^k)k\right] \\
&= r\left[1 + (1-(1-p)^k)k\right] \\
&= \frac{N}{k}\left[1 + (1-(1-p)^k)k\right] \\
&= N\left(\frac{1}{k} + 1 - (1-p)^k\right).
\end{aligned}$$

3. The expected number of tests is at most N is the equation

$$\begin{aligned}
N\left(\frac{1}{k} + 1 - (1-p)^k\right) &\leq N \\
\frac{1}{k} + 1 - (1-p)^k &\leq 1 \\
\frac{1}{k} &\leq (1-p)^k \\
\left(\frac{1}{k}\right)^{\frac{1}{k}} &\leq 1 - p \\
\frac{1}{1-p} &\leq k^{\frac{1}{k}}.
\end{aligned}$$

The maximum of $k^{\frac{1}{k}}$ arises where $k = 3$, and this is valid since $k = 3 \mid N$. Hence,

$$\begin{aligned}
\frac{1}{1-p} &\leq 3^{\frac{1}{3}} \\
p &\leq 1 - 3^{-\frac{1}{3}},
\end{aligned}$$

and hence such largest value of p is

$$p = 1 - 3^{-\frac{1}{3}}.$$

Notice that

$$\begin{aligned}
1 - 3^{-\frac{1}{3}} > \frac{1}{4} &\iff \frac{3}{4} > 3^{-\frac{1}{3}} \\
&\iff \left(\frac{3}{4}\right)^3 > 3^{-1} \\
&\iff \frac{27}{64} > \frac{1}{3} \\
&\iff 81 > 64,
\end{aligned}$$

which is true, and so this value of p is greater than $\frac{1}{4}$.

4. We would like to show that if $pk \ll 1$, then $1 - (1-p)^k \approx pk$.
Notice that

$$\begin{aligned}
1 - (1-p)^k &= 1 - \sum_{i=0}^{k} \binom{k}{i} (-p)^k \\
&= 1 - (1 - kp + \cdots) \\
&\approx kp,
\end{aligned}$$

and hence

$$\mathrm{E}(X) = N\left(\frac{1}{k} + 1 - (1-p)^k\right) \approx N\left(\frac{1}{k} + pk\right).$$

If $p = 0.01$, $k = 10$, we have

$$\mathrm{E}(X) \approx N\left(\frac{1}{10} + 0.01 \cdot 10\right) = N \cdot \frac{2}{10} = \frac{1}{5}N,$$

which is 20% of N.

2024.2 Question 12

1. Let X_i be the number that the ith player receives, and let Ada be the first player. We have

$$\begin{aligned}\mathrm{P}(X_1 = a, X_2 > X_1, X_3 > X_1, \cdots, X_k > X_1) &= \mathrm{P}(X_1 = a, X_2 > a, X_3 > a, \cdots, X_k > a)\\ &= \mathrm{P}(X_1 = a)\,\mathrm{P}(X_2 > a)\,\mathrm{P}(X_3 > a)\cdots\mathrm{P}(X_k > a)\\ &= \frac{1}{n}\cdot\frac{n-a}{n}\cdot\frac{n-a}{n}\cdots\frac{n-a}{n}\\ &= \frac{(n-a)^{k-1}}{n^k}.\end{aligned}$$

Hence, the probability of Ada winning this is

$$\begin{aligned}\mathrm{P}(X_2 > X_1, X_3 > X_1, \cdots, X_k > X_1) &= \sum_{a=1}^{n-1}\mathrm{P}(X_1 = a, X_2 > X_1, X_3 > X_1, \cdots, X_k > X_1)\\ &= \sum_{a=1}^{n-1}\frac{(n-a)^{k-1}}{n^k}\\ &= \frac{1}{n^k}\sum_{a=1}^{n-1}a^{k-1},\end{aligned}$$

and the probability of there being a winner is the sum of the probabilities of each player winning, which are all equal to the probability of Ada winning by symmetry, and hence is equal to

$$k\cdot\frac{1}{n^k}\sum_{a=1}^{n-1}a^{k-1} = \frac{k}{n^k}\sum_{a=1}^{n-1}a^{k-1}.$$

If $k = 4$, then this probability is given by

$$\begin{aligned}\mathrm{P} &= \frac{4}{n^4}\sum_{a=1}^{n-1}a^3\\ &= \frac{4}{n^4}\cdot\frac{(n-1)^2n^2}{4}\\ &= \frac{(n-1)^2}{n^2},\end{aligned}$$

precisely as desired.

2. Similarly, let X_i be the number that the ith player receives, and let Ada be the first player, and Bob be the second player. We have

$$\begin{aligned}&\mathrm{P}(X_1 = a, X_2 = a+d+1, X_1 < X_3 < X_2, \cdots, X_1 < X_k < X_2)\\ &= \mathrm{P}(X_1 = a, X_2 = a+d+1, a < X_3 < a+d+1, \cdots, a < X_k < a+d+1)\\ &= \mathrm{P}(X_1 = a)\,\mathrm{P}(X_2 = a+d+1)\,\mathrm{P}(a < X_3 < a+d+1)\cdots\mathrm{P}(a < X_k < a+d+1)\\ &= \frac{1}{n}\cdot\frac{1}{n}\cdot\frac{d}{n}\cdots\frac{d}{n}\\ &= \frac{d^{k-2}}{n^k}.\end{aligned}$$

Hence, the probability that both Ada and Bob winning this is

$$
\begin{aligned}
&\mathrm{P}(X_1 < X_3 < X_2, \cdots, X_1 < X_k < X_2) \\
={}& \sum_{d=1}^{n-2} \sum_{a=1}^{n-d-1} \mathrm{P}(X_1 = a, X_2 = a + d + 1, X_1 < X_3 < X_2, \cdots, X_1 < X_k < X_2) \\
={}& \sum_{d=1}^{n-2} \sum_{a=1}^{n-d-1} \frac{d^{k-2}}{n^k} \\
={}& \sum_{d=1}^{n-2} \frac{(n-d-1)d^{k-2}}{n^k} \\
={}& \frac{1}{n^k} \sum_{d=1}^{n-2} (n-d-1)d^{k-2} \\
={}& \frac{1}{n^k} \left[(n-1) \sum_{d=1}^{n-2} d^{k-2} - \sum_{d=1}^{n-2} d^{k-1} \right].
\end{aligned}
$$

Hence, the probability that there are two winners in this game is the sum of the probabilities of each ordered pair of players winning (since there is one winning by having a bigger number, and one winning by having a smaller number), and hence is equal to

$$
2 \cdot \binom{k}{2} \cdot \frac{1}{n^k} \left[(n-1) \sum_{d=1}^{n-2} d^{k-2} - \sum_{d=1}^{n-2} d^{k-1} \right].
$$

When $k = 4$, the probability is

$$
\begin{aligned}
\mathrm{P} &= 2 \cdot \binom{4}{2} \cdot \frac{1}{n^4} \left[(n-1) \sum_{d=1}^{n-2} d^2 - \sum_{d=1}^{n-2} d^3 \right] \\
&= 2\dot{6} \cdot \frac{1}{n^4} \left[\frac{(n-1)(n-2)(n-1)(2n-3)}{6} - \frac{(n-2)^2(n-1)^2}{4} \right] \\
&= 12 \cdot \frac{1}{n^4} \cdot (n-1)^2(n-2) \left[\frac{2(2n-3) - 3(n-2)}{12} \right] \\
&= \frac{(n-1)^2(n-2)}{n^4} \cdot n \\
&= \frac{(n-2)(n-1)^2}{n^3}.
\end{aligned}
$$

The probability of there being a winner due to having the biggest number (denote this event as B), is the same as there being a winner due to having the lowest number (denote this event as L), which are both equal to the answer to the first part of the question:

$$
\mathrm{P}(B) = \mathrm{P}(L) = \frac{(n-1)^2}{n^2}.
$$

The event of having two winners is B, L and the event of having precisely one winner is $B, \bar{L}$ or $L, \bar{B}$. By the inclusion-exclusion principle, the probability of having precisely one winner is given by

$$
\begin{aligned}
\mathrm{P} &= \mathrm{P}(B) + \mathrm{P}(L) - 2\,\mathrm{P}(B, L) \\
&= 2 \cdot \frac{(n-1)^2}{n^2} - 2 \cdot \frac{(n-2)(n-1)^2}{n^3} \\
&= \frac{2(n-1)^2}{n^3} \cdot [n - (n-2)] \\
&= \frac{4(n-1)^2}{n^3}.
\end{aligned}
$$

This probability is smaller than $\mathrm{P}(B, L)$, if and only if

$$\begin{aligned} \frac{4(n-1)^2}{n^3} &< \frac{(n-2)(n-1)^2}{n^3} \\ 4 &< n-2 \\ n &> 6, \end{aligned}$$

and hence the minimum value of n for this is 7.