2023 Paper 3

2023.3 Question 1

1. The line through P and Q has gradient

$$\frac{aq^2 - ap^2}{2aq - 2ap} = \frac{q^2 - p^2}{2(q-p)} = \frac{p+q}{2},$$

and so it has equation

$$\begin{aligned} y - ap^2 &= \frac{1}{2}(p+q)(x - 2ap) \\ y &= \frac{1}{2}(p+q)x + ap^2 - ap^2 - apq \\ y &= \frac{1}{2}(p+q)x - apq. \end{aligned}$$

The line is tangent to the circle with centre $(0, 3a)$ and radius $2a$, if and only if its distance from $(0, 3a)$ is $2a$.

The line has equation

$$2y - (p+q)x + 2apq = 0$$

and hence the distance is

$$\begin{aligned} d &= \frac{|2 \cdot 3a - (p+q) \cdot 0 + 2apq|}{\sqrt{2^2 + (p+q)^2}} \\ &= \frac{|6a + 2apq|}{\sqrt{4 + p^2q^2 + 6pq + 5}} \\ &= \frac{|2a(3+pq)|}{\sqrt{(pq+3)^2}} \\ &= \frac{2a|3+pq|}{|3+pq|} \\ &= 2a, \end{aligned}$$

and so the distance from l to $(0, 3a)$ is $2a$ as desired.

2. We rearrange the condition to an equation in q

$$\begin{aligned} p^2 + 2pq + q^2 &= p^2q^2 + 6pq + 5 \\ \left(p^2 - 1\right) q^2 + 4pq + \left(5 - p^2\right) &= 0, \end{aligned}$$

and since $p^2 \neq 1$, this must be a quadratic.

We examine the discriminant, Δ:

$$\begin{aligned} \Delta &= (4p)^2 - 4\left(p^2 - 1\right)\left(5 - p^2\right) \\ &= 16p^2 - 4\left(-p^4 - 5 + 6p^2\right) \\ &= 4p^4 - 8p^2 + 20 \\ &= 4\left(p^4 - 4p^2 + 5\right) \\ &= 4\left[\left(p^2 - 2\right)^2 + 1\right] \\ &\geq 4 \cdot 1 \\ &= 4 \\ &> 0, \end{aligned}$$

and so $\Delta > 0$, meaning there will be two distinct real values of q satisfying the condition.

By Vieta's Theorem, we have $q_1 + q_2 = -\frac{4p}{p^2-1}$, and $q_1 q_2 = \frac{5-p^2}{p^2-1}$.

3. Notice that

$$(q_1 + q_2)^2 = \frac{16p^2}{(p^2-1)^2},$$

and

$$\begin{aligned}
q_1^2 q_2^2 + 6q_1q_2 + 5 &= \frac{(5-p^2)^2}{(p^2-1)^2} + \frac{6 \cdot (5-p^2)}{(p^2-1)} + 5 \\
&= \frac{(5-p^2)^2 + 6(5-p^2)(p^2-1) + 5(p^2-1)^2}{(p^2-1)^2} \\
&= \frac{25 - 10p^2 + p^4 - 6p^4 + 36p^2 - 30 + 5p^4 - 10p^2 + 5}{(p^2-1)^2} \\
&= \frac{16p^2}{(p^2-1)^2},
\end{aligned}$$

and so $(q_1 + q_2)^2 = q_1^2 q_2^2 + 6q_1q_2 + 5$.

Let $P\left(2ap, ap^2\right)$ for some $p \neq 1$, and let the corresponding solutions to the condition be q_1, q_2. Define the points $Q_1\left(2aq_1, aq_1^2\right)$ and $Q_2\left(2aq_2, aq_2^2\right)$.

The previous part of the question shows that Q_1 and Q_2 exists and are distinct.

The first part ensures that PQ_1 and PQ_2 are tangents to the circle.

But since q_1 and q_2 satisfies the conditions as well, we must have Q_1Q_2 being a tangent to the circle as well.

Hence, triangle PQ_1Q_2 has all vertices on $x^2 = 4ay$, and that all three sides are tangent to the desired circle.

2023.3 Question 2

1. If the two curves meet at $\theta = \alpha$, then α must satisfy that

$$k(1 + \sin\alpha) = k + \cos\alpha.$$

Subtracting k on both sides, we have

$$k \sin\alpha = \cos\alpha,$$

and since $k > 1 > 0$, $\sin\alpha$ and $\cos\alpha$ cannot be simultaneously zero, they must both be non-zero. Dividing through both sides by $\cos\alpha$ gives

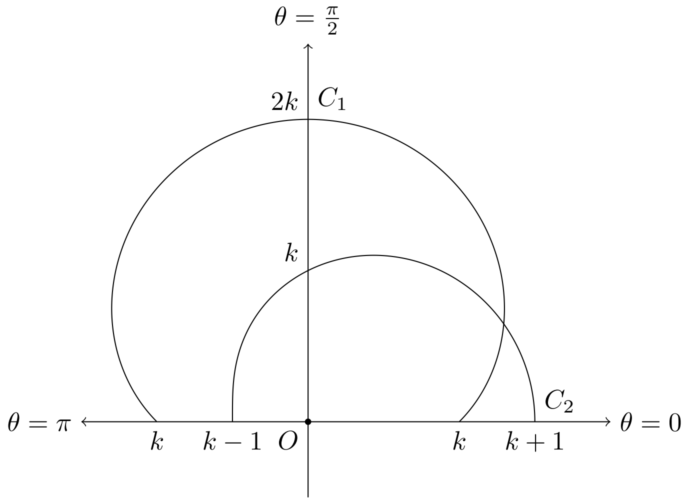
$$k \tan\alpha = 1$$

and hence

$$\tan\alpha = \frac{1}{k}$$

as desired.

The curves are as follows.



2. The area of A is given by

$$\begin{aligned}
[A] &= \frac{1}{2} \cdot \int_0^\alpha (k(1 + \sin\theta))^2 \,\mathrm{d}\theta \\
&= \frac{k^2}{2} \cdot \int_0^\alpha \left(1 + 2\sin\theta + \sin^2\theta\right) \mathrm{d}\theta \\
&= \frac{k^2}{2} \cdot \int_0^\alpha \left(1 + 2\sin\theta + \frac{1 - \cos 2\theta}{2}\right) \mathrm{d}\theta \\
&= \frac{k^2}{2} \cdot \left[\frac{3}{2} \cdot \theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^\alpha \\
&= \frac{k^2}{2} \cdot \left[\left(\frac{3}{2}\alpha - 2\cos\alpha - \frac{1}{4}\sin 2\alpha\right) - \left(0 - 2 - \frac{1}{4} \cdot 0\right)\right] \\
&= \frac{k^2}{2}\left[\frac{3}{2}\alpha - 2\cos\alpha - \frac{1}{2}\sin\alpha\cos\alpha + 2\right] \\
&= \frac{k^2}{4}\left(3\alpha - \sin\alpha\cos\alpha\right) + k^2\left(1 - \cos\alpha\right).
\end{aligned}$$

3. The area of B is given by

$$\begin{aligned}
[B] &= \frac{1}{2} \cdot \int_{\alpha}^{\pi} (k + \cos\theta)^2 \, \mathrm{d}\theta \\
&= \frac{1}{2} \cdot \int_{\alpha}^{\pi} \left(k^2 + 2k\cos\theta + \cos^2\theta\right) \mathrm{d}\theta \\
&= \frac{1}{2} \cdot \int_{\alpha}^{\pi} \left(k^2 + 2k\cos\theta + \frac{1+\cos 2\theta}{2}\right) \mathrm{d}\theta \\
&= \frac{1}{2} \cdot \left[\left(k^2 + \frac{1}{2}\right)\theta + 2k\sin\theta + \frac{\sin 2\theta}{4}\right]_{\alpha}^{\pi} \\
&= \frac{1}{2} \cdot \left[\left(\left(k^2 + \frac{1}{2}\right)\pi + 2k \cdot 0 + \frac{0}{4}\right) - \left(\left(k^2 + \frac{1}{2}\right)\alpha + 2k \cdot \sin\alpha + \frac{\sin 2\alpha}{4}\right)\right] \\
&= \frac{1}{2} \cdot \left[\left(k^2 + \frac{1}{2}\right)(\pi - \alpha) - 2k\sin\alpha - \frac{\sin\alpha\cos\alpha}{2}\right] \\
&= \frac{1}{4} \cdot \left(2k^2\pi + \pi - 2k^2\alpha - \alpha - 4k\sin\alpha - \sin\alpha\cos\alpha\right).
\end{aligned}$$

4. T is given by

$$\begin{aligned}
T &= \frac{1}{2} \cdot \int_{0}^{\pi} (k + \cos\theta)^2 \, \mathrm{d}\theta \\
&= \frac{1}{2} \cdot \left[\left(k^2 + \frac{1}{2}\right)\theta + 2k\sin\theta + \frac{\sin 2\theta}{4}\right]_{0}^{\pi} \\
&= \frac{1}{4} \cdot \left(2k^2\pi + \pi - 2k^2 \cdot 0 - 0 - 4k \cdot 0 - 0 \cdot 1\right) \\
&= \frac{\pi\left(2k^2 + 1\right)}{4}.
\end{aligned}$$

As $k \to \infty$, $\frac{1}{k} = \tan\alpha \to 0^+$, and therefore,

$$\alpha, \sin\alpha, \tan\alpha \approx \frac{1}{k}$$

and

$$\cos\alpha \approx 1 - \frac{1}{2k^2}.$$

Therefore, considering only terms with the highest power of k

$$\begin{aligned}
[A] &= \frac{k^2}{4}(3\alpha - \sin\alpha\cos\alpha) + k^2(1 - \cos\alpha) \\
&\approx \frac{k^2}{4}\left(3\left(\frac{1}{k}\right) - \left(\frac{1}{k}\right)\left(1 - \frac{1}{2k^2}\right)\right) + k^2\left(1 - \left(1 - \frac{1}{2k^2}\right)\right) \\
&= \frac{k^2}{4}\left(\frac{2}{k} + \frac{1}{2k^3} + 1\right) \\
&\approx \frac{k}{2},
\end{aligned}$$

and

$$\begin{aligned}
[B] &= \frac{1}{4} \cdot \left(2k^2\pi + \pi - 2k^2\alpha - \alpha - 4k\sin\alpha - \sin\alpha\cos\alpha\right) \\
&= \frac{1}{4} \cdot \left(2k^2\pi + \pi - 2k^2 \cdot \frac{1}{k} - \frac{1}{k} - 4k \cdot \frac{1}{k} - \frac{1}{k} \cdot \left(1 - \frac{1}{2k^2}\right)\right) \\
&= \frac{1}{4} \cdot \left(2k^2\pi + \pi - 2k - \frac{1}{k} - 4 - \frac{1}{k} + \frac{1}{2k^3}\right) \\
&\approx \frac{k^2\pi}{2}.
\end{aligned}$$

Therefore,

$$R = [A] + [B] \approx \frac{k^2\pi}{2}$$

Hence,

$$\begin{aligned}\lim_{k\to\infty} \frac{R}{T} &= \lim_{k\to\infty} \frac{\frac{k^2\pi}{2}}{\frac{\pi(2k^2+1)}{4}} \\ &= \lim_{k\to\infty} \frac{2k^2}{2k^2+1} \\ &= 1\end{aligned}$$

as desired.

Similarly, S is given by

$$\begin{aligned}S &= \frac{1}{2} \cdot \int_0^\pi (k(1+\sin\theta))^2 \, \mathrm{d}\theta \\ &= \frac{k^2}{2} \cdot \left[\frac{3}{2} \cdot \theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^\pi \\ &= \frac{k^2}{4}(3\pi - \sin\pi\cos\pi) + k^2(1-\cos\pi) \\ &= \frac{k^2}{4} \cdot 3\pi + 2k^2 \\ &= \left(2 + \frac{3\pi}{4}\right)k^2.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{k\to\infty} \frac{R}{S} &= \lim_{k\to\infty} \frac{\frac{k^2\pi}{2}}{\left(2+\frac{3\pi}{4}\right)k^2} \\ &= \lim_{k\to\infty} \frac{4\pi}{8+3\pi}.\end{aligned}$$

2023.3 Question 3

1. We consider the distance between $a \pm sbi$ and $a + b$. We have

$$\begin{aligned}|(a \pm sbi) - (a+b)| &= |\pm sbi - b| \\ &= |(-1 \pm si)b| \\ &= |-1 \pm si||b| \\ &= \sqrt{1+s^2}|b|\end{aligned}$$

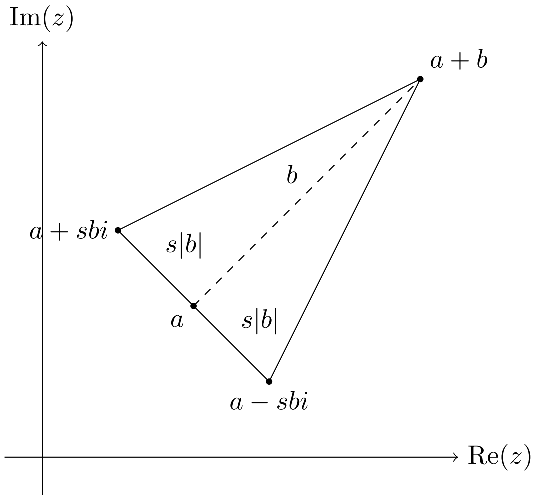
is independent of the plus or minus sign. This shows that the distances from points $a \pm sbi$ to $a+b$ are equal, and this means that the three points form an isosceles triangle.

Notice that the midpoint of $a + sbi$ and $a - sbi$ is a. Therefore, for any isosceles triangle in the complex plane, the midpoint of the base of the isosceles triangle is represented by the complex number a, and the vector from the midpoint of the base to the top vertex is represented by the complex number b.

Notice that

$$\sqrt{1+s^2}|b| = \sqrt{|b|^2 + (s|b|)^2}$$

and this means s is the ratio of the length of half the base to the length of the height (represented by b).



2. From the previous part, three points in the complex plane represent an isosceles triangle if and only if they could be represented as $a + sbi$, $a - sbi$ and $a + b$ for some complex a, b where $b \neq 0$ and positive real s (and such representation is unique).

Let z_1, z_2 and z_3 be the roots of this equation. We notice that from Vieta's Theorem,

$$\begin{cases} z_1 + z_2 + z_3 = 0, \\ z_1z_2 + z_1z_3 + z_2z_3 = p, \\ z_1z_2z_3 = -q. \end{cases}$$

On the other hand,

$$z_1 + z_2 + z_3 = (a+sbi) + (a-sbi) + (a+b) = 3a + b,$$

$$\begin{aligned}z_1z_2 + z_1z_3 + z_2z_3 &= (a+sbi)(a-sbi) + (a+sbi)(a+b) + (a-sbi)(a+b) \\ &= a^2 + s^2b^2 + 2a(a+b) \\ &= 3a^2 + 2ab + s^2b^2,\end{aligned}$$

and

$$\begin{aligned} z_1z_2z_3 &= (a+sbi)(a-sbi)(a+b) \\ &= (a^2+s^2b^2)(a+b) \\ &= a^3+a^2b+s^2ab^2+s^2b^3. \end{aligned}$$

Since $3a+b=0$, we have $b=-3a$, and therefore

$$\begin{aligned} p &= 3a^2+2ab+s^2b^2 \\ &= 3a^2+2a(-3a)+s^2(-3a)^2 \\ &= 3a^2-6a^2+s^2\cdot 9a^2 \\ &= 9s^2a^2-3a^2 \\ &= 3a^2\left(3s^2-1\right), \end{aligned}$$

and

$$\begin{aligned} q &= -\left(a^3+a^2b+s^2ab^2+s^2b^3\right) \\ &= -\left(a^3+a^2(-3a)+s^2a(-3a)^2+s^2(-3a)^3\right) \\ &= -\left(a^3-3a^3+9s^2a^3-27s^2a^3\right) \\ &= -\left(-2a^3-18s^2a^3\right) \\ &= 2a^3\left(9s^2+1\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{p^3}{q^2} &= \frac{\left(3a^2\left(3s^2-1\right)\right)^3}{\left(2a^3\left(9s^2+1\right)\right)^2} \\ &= \frac{27a^6\left(3s^2-1\right)^3}{4a^6\left(9s^2+1\right)^2} \\ &= \frac{27\left(3s^2-1\right)^3}{4\left(9s^2+1\right)^2} \end{aligned}$$

for this value of s of the isosceles triangle, showing precisely that such s does exist.

3. This function is defined for $x \neq -\frac{1}{9}$. Within the domain, it is positive for $x > \frac{1}{3}$ and negative for $x < \frac{1}{3}$. Therefore, as $x \to -\frac{1}{9}$, $y \to -\infty$.

   As $x \to \pm\infty$, we find the asymptote by long division. We have

$$\begin{aligned} y &= \frac{(3x-1)^3}{(9x+1)^2} \\ &= \frac{27x^3-27x^2+9x-1}{81x^2+18x+1} \\ &= \frac{1}{3}x+\frac{-33x^2+\frac{26}{3}x-1}{81x^2+18x+1} \\ &= \frac{1}{3}x-\frac{11}{27}+\frac{16x-\frac{16}{27}}{81x^2+18x+1} \end{aligned}$$

   and hence $y=\frac{1}{3}x-\frac{11}{27}$ is an asymptote as $x \to \pm\infty$.

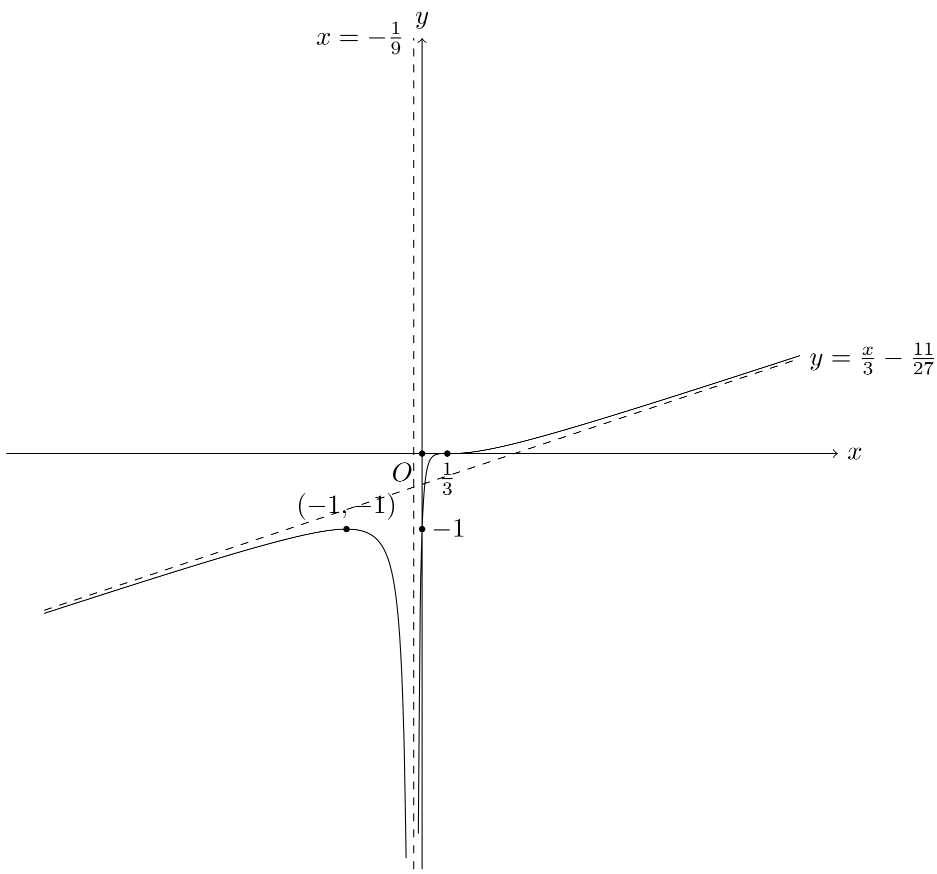
Differentiating this gives us

$$\begin{aligned} y' &= \frac{\left[(3x-1)^3\right]'(9x+1)^2 - \left[(9x+1)^2\right]'(3x-1)^3}{(9x+1)^4} \\ &= \frac{3\cdot 3\cdot (3x-1)^2(9x+1)^2 - 2\cdot 9\cdot (9x+1)\cdot (3x-1)^3}{(9x+1)^4} \\ &= \frac{9(3x-1)^2(9x+1) - 18(3x-1)^3}{(9x+1)^3} \\ &= \frac{9(3x-1)^2}{(9x+1)^3}\left[(9x+1) - 2(3x-1)\right] \\ &= \frac{27(3x-1)^2(x+1)}{(9x+1)^3}. \end{aligned}$$

Therefore, $y' = 0$ if and only if $x = \frac{1}{3}$ (which is also a zero), or $x = -1$ (which corresponds to $y = \frac{(3\cdot(-1)-1)^3}{(9\cdot(-1)+1)^2} = \frac{-64}{64} = -1$, which is $(-1,-1)$).

The y-intercept is -1.

Therefore, the graph is as follows.



4. We have shown in part (ii) that such s exists and could be the corresponding s for the isosceles triangle represented by the three roots. Therefore, the ratio is real.

   Furthermore,

$$\frac{p^3}{q^2} = \frac{27\left(3s^2-1\right)^3}{4\left(9s^2+1\right)^2} = \frac{27}{4}\cdot \left.\frac{(3x-1)^3}{(9x+1)^2}\right|_{x=s^2}.$$

   Therefore, since the minimum of $y = \frac{(3x-1)^3}{(9x+1)^2}$ for $x \geq 0$ is at $y = -1$ when $x = 0$, we must have

$$\frac{p^3}{q^2} \geq \frac{27}{4}\cdot(-1) = -\frac{27}{4}$$

as desired.

2023.3 Question 4

1. Consider $z = \exp(i\theta)$. On one hand, we have

$$z^{2n+1} = \exp(i(2n+1)\theta) = \cos(2n+1)\theta + i\sin(2n+1)\theta$$

and on the other hand,

$$\begin{aligned} z^{2n+1} &= (\cos\theta + i\sin\theta)^{2n+1} \\ &= \sum_{k=0}^{2n+1} \binom{2n+1}{k} \cos^{2n+1-k}\theta \sin^k\theta \cdot i^k. \end{aligned}$$

Taking the real part on both sides, we notice that even ks produce a real term for the sum, and odd ks produce an imaginary term for the sum. Hence,

$$\begin{aligned} \cos(2n+1)\theta &= \sum_{r=0}^{n} \binom{2n+1}{2r} \cos^{2n+1-2r}\theta \sin^{2r}\theta \cdot i^{2r} \\ &= \sum_{r=0}^{n} \binom{2n+1}{2r} \cos^{2n+1-2r}\theta \sin^{2r}\theta \cdot (-1)^r \\ &= \sum_{r=0}^{n} \binom{2n+1}{2r} \cos^{2n+1-2r}\theta \left(-\sin^2\theta\right)^r \\ &= \sum_{r=0}^{n} \binom{2n+1}{2r} \cos^{2n+1-2r}\theta \left(\cos^2\theta - 1\right)^r, \end{aligned}$$

as desired.

2. From the previous part, we can conclude that for $-1 \le x \le 1$, we have

$$p(x) = 1 + \sum_{r=0}^{n} \binom{2n+1}{2r} x^{2n+1-2r} \left(x^2 - 1\right)^r$$

and this must be the expression for $p(x)$ for all real numbers x.

Further simplification yields

$$\begin{aligned} p(x) &= 1 + \sum_{r=0}^{n} \binom{2n+1}{2r} x^{2n+1-2r} \sum_{k=0}^{r} \binom{r}{k} x^{2k} (-1)^{r-k} \\ &= 1 + \sum_{r=0}^{n} \sum_{k=0}^{r} \binom{2n+1}{2r} \binom{r}{k} (-1)^{r-k} x^{2n+1+2k-2r}. \end{aligned}$$

For the coefficient of x^{2n+1}, it must be the case that $k = r$ for the contribution of the coefficient, and hence it is equal to

$$\sum_{r=0}^{n} \binom{2n+1}{2r} \binom{r}{r} (-1)^{r-r} = \sum_{r=0}^{n} \binom{2n+1}{2r}.$$

We consider the expansion of $(1+t)^{2n+1}$. By the binomial theorem, we have

$$(1+t)^{2n+1} = \sum_{r=0}^{2n+1} \binom{2n+1}{r} t^r = \sum_{r=0}^{n} \binom{2n+1}{2r} t^{2r} + \sum_{r=0}^{n} \binom{2n+1}{2r+1} t^{2r+1}.$$

Let $t = 1$, and we have

$$2^{2n+1} = \sum_{r=0}^{n} \binom{2n+1}{2r} + \sum_{r=0}^{n} \binom{2n+1}{2r+1}.$$

Let $t = -1$, and we have

$$0 = \sum_{r=0}^{n} \binom{2n+1}{2r} (-1)^{2r} + \sum_{r=0}^{n} \binom{2n+1}{2r+1} (-1)^{2r+1}$$

and hence

$$0 = \sum_{r=0}^{n} \binom{2n+1}{2r} - \sum_{r=0}^{n} \binom{2n+1}{2r+1}.$$

Let $A = \sum_{r=0}^{n} \binom{2n+1}{2r}$, and $B = \sum_{r=0}^{n} \binom{2n+1}{2r+1}$. We have $A + B = 2^{2n+1}$ and $A - B = 0$, giving $A = B = 2^{2n}$.

Therefore, the coefficient of x^{2n+1} in the polynomial $p(x)$ is A, which is 2^{2n} as desired.

3. We recall that

$$p(x) = 1 + \sum_{r=0}^{n} \sum_{k=0}^{r} \binom{2n+1}{2r} \binom{r}{k} (-1)^{r-k} x^{2n+1+2k-2r}.$$

For $2n + 1 + 2k - 2r = 2n - 1$, it must be the case that $k = r - 1$, and therefore the coefficient is given by

$$\sum_{r=0}^{n} \binom{2n+1}{2r} \binom{r}{r-1} (-1)^{r-(r-1)} = -\sum_{r=0}^{n} r \binom{2n+1}{2r}.$$

What remains is to show that

$$\sum_{r=0}^{n} r \binom{2n+1}{2r} = (2n+1) 2^{2n-2}.$$

Notice that from the definition of the binomial coefficient

$$\begin{aligned}
\sum_{r=0}^{n} r \binom{2n+1}{2r} &= \frac{1}{2} \sum_{r=1}^{n} 2r \binom{2n+1}{2r} \\
&= \frac{1}{2} \sum_{r=1}^{n} 2r \cdot \frac{(2n+1)!}{(2n+1-2r)!(2r)!} \\
&= \frac{1}{2} \sum_{r=1}^{n} \frac{(2n+1)!}{(2n+1-2r)!(2r-1)!} \\
&= \frac{2n+1}{2} \sum_{r=1}^{n} \frac{(2n)!}{(2n+1-2r)!(2r-1)!} \\
&= \frac{2n+1}{2} \sum_{r=1}^{n} \binom{2n}{2r-1} \\
&= \frac{2n+1}{2} \sum_{r=0}^{n-1} \binom{2n}{2r+1}.
\end{aligned}$$

Similar to the previous part, consider

$$(1+t)^{2n} = \sum_{r=0}^{2n} \binom{2n}{r} t^r = \sum_{r=0}^{n} \binom{2n}{2r} t^{2r} + \sum_{r=0}^{n-1} \binom{2n}{2r+1} t^{2r+1}.$$

Let $t = 1$, and we have

$$2^{2n} = \sum_{r=0}^{n} \binom{2n}{2r} + \sum_{r=0}^{n-1} \binom{2n}{2r+1}.$$

Let $t = -1$, and we have

$$2^{2n} = \sum_{r=0}^{n} \binom{2n}{2r} - \sum_{r=0}^{n-1} \binom{2n}{2r+1}.$$

Therefore, we have

$$\sum_{r=0}^{n-1} \binom{2n}{2r+1} = 2^{2n-1}.$$

Hence,

$$\begin{aligned} \frac{2n+1}{2} \sum_{r=0}^{n-1} \binom{2n}{2r+1} &= \frac{2n+1}{2} \cdot 2^{2n-1} \\ &= (2n+1)2^{2n-2}, \end{aligned}$$

and therefore the coefficient is given by $-(2n+1)2^{2n-2}$, as desired.

4. The coefficient of x^n in $q(x)$ must be 2^n (to contribute to the x^{2n+1} term in $p(x)$).

   Let a_k be the coefficient of x^k in $q(x)$.

   The term x^{2n} in $p(x)$ has zero as its coefficient, since $2n+1+2k-2r$ is always odd. It must be given by x multiplied by some term with power x^{2n-1} in $q(x)^2$, which is $x^n \cdot x^{n-1}$ or $x^{n-1} \cdot x^n$, or 1 multiplied by some term with power x^{2n}, which must be $x^n \cdot x^n$. Therefore,

$$0 = 2a_n a_{n-1} + a_n^2,$$

   and hence

$$a_{n-1} = -\frac{a_n}{2} = -2^{n-1}.$$

   The term x^{2n-1} in $p(x)$ is given by x multiplied by some term with power x^{2n-2} in $q(x)^2$, which is $x^n \cdot x^{n-2}$, $x^{n-1} \cdot x^{n-1}$ or $x^{n-2} \cdot x^n$, or 1 multiplied by some term with power x^{2n-1} in $q(x)^2$, which is $x^n \cdot x^{n-1}$ or $x^{n-1} \cdot x^n$. Therefore,

$$-(2n+1)2^{2n-2} = 2a_n a_{n-2} + a_{n-1}^2 + 2a_n a_{n-1},$$

   and hence

$$-(2n+1)2^{2n-2} = 2 \cdot 2^n \cdot a_{n-2} + 2^{2n-2} - 2 \cdot 2^n \cdot 2^{n-1},$$

   which means

$$-(2n+1)2^{n-3} = a_{n-2} + 2^{n-3} - 2^{n-1}.$$

   Hence,

$$\begin{aligned} a_{n-2} &= 2^{n-1} - 2^{n-3} - (2n+1)2^{n-3} \\ &= 2^{n-1} - (1 + (2n+1))2^{n-3} \\ &= 2^{n-1} - (2n+2)2^{n-3} \\ &= 2^{n-1} - 2(n+1)2^{n-3} \\ &= 2^{n-1} - (n+1)2^{n-2} \\ &= (2 - (n+1))2^{n-2} \\ &= (1-n)2^{n-2}, \end{aligned}$$

   which means the coefficient of x^{n-2} in $q(x)$ is $2^{n-2}(1-n)$, as desired.

2023.3 Question 5

1. If x, y are both non-zero,

$$\begin{aligned}\frac{1}{x}+\frac{2}{y}&=\frac{2}{7}\\7y+2\cdot 7x&=2xy\\2xy-14x-7y&=0\\2xy-14x-7y+49&=49\\2x(y-7)-7(y-7)&=49\\(2x-7)(y-7)&=49.\end{aligned}$$

We must have $2x-7\geq 2\cdot 1-7=-5$ and $y-7\geq 1-7=-6$.

$2x-7$ and $y-7$ are both integers, and we do casework considering expressing 49 into a product of two integers that are both not less than -6.

- $49=1\times 49$, $2x-7=1$ and $y-7=49$, giving us $(x,y)=(4,56)$.
- $49=7\times 7$, $2x-7=7$ and $y=7=7$, giving us $(x,y)=(7,14)$.
- $49=49\times 1$, $2x-7=49$ and $x-7=1$, giving us $(x,y)=(28,8)$.

Since all x, y are non-zero, we can conclude that the solutions are $(x,y)=(4,56),(7,14),(28,8)$.

2. We have

$$\begin{aligned}p^2+pq+q^2&=n^2\\p^2+2pq+q^2&=n^2+pq\\(p+q)^2&=n^2+pq\\(p+q)^2-n^2&=pq\\(p+q+n)(p+q-n)&=pq.\end{aligned}$$

We must have $p+q+n>p+q-n$ since n is a positive integer. We have $p+q+n>p,q>1>0$. It must be the case that $p+q-n$ is positive as well.

Therefore, $p+q+n$ cannot be $1,p,q$, and it must be the case that $p+q+n=pq$ and $p+q-n=1$.

Therefore, $p+q=n+1$, and $pq=p+q+n=2n+1$.

Hence, p, q are solutions to the quadratic equation in t

$$t^2-(n+1)t+(2n+1)=0.$$

Solving this gives us

$$\begin{aligned}p,q&=\frac{(n+1)\pm\sqrt{(n+1)^2-4\cdot(2n+1)}}{2}\\&=\frac{(n+1)\pm\sqrt{n^2-6n-3}}{2}.\end{aligned}$$

We have $n^2-6n-3=(n-3)^2-12$ must be a perfect square for p,q to be rational (and they are since all integers are rational).

Consider $a,b\geq 0$, $a,b\in\mathbb{N}$ such that $a^2-b^2=(a+b)(a-b)=12$.

$a+b$ and $a-b$ must take the same odd-even parity, and the only possibility is therefore $a+b=6$ and $a-b=2$, solving to $(a,b)=(4,2)$.

Therefore, $n-3=4$, $n=7$, and we solve for

$$p,q=\frac{8\pm\sqrt{49-42-3}}{2}=4\pm 1$$

and $(p,q)=(3,5),(5,3)$ are indeed primes, and $n=7$.

3. If $p + q - n \geq p$, then $q \geq n$, and for the original equation,

$$\text{LHS} = p^3 + q^3 + 3pq^2 > q^3 \geq n^3 = \text{RHS},$$

and hence LHS > RHS is impossible. Hence, $p + q - n < p$.

It must also be the case for $p + q - n < q$.

We have

$$\begin{aligned} p^3 + q^3 + 3pq^2 &= n^3 \\ p^3 + q^3 + 3pq^2 + 3p^2q &= n^3 + 3p^2q \\ (p+q)^3 &= n^3 + 3p^2q \\ (p+q)^3 - n^3 &= 3p^2q \\ (p+q-n)\left[(p+q)^2 + (p+q)\cdot n + n^2\right] &= 3p^2q. \end{aligned}$$

The factors of $3p^2q$ are (given p and q are prime),

$$1, 3, p, q, 3p, 3q, p^2, pq, 3p^2, 3pq, p^2q, 3p^2q,$$

and since $p + q - n < p$ and $p + q - n < q$, it must be either the case that $p + q - n = 1$ or $p + q - n = 3$.

- If $p + q - n = 1$, then $p + q = n + 1$, we have

$$\begin{aligned} (p+q)^2 + (p+q)n + n^2 &= 3p^2q \\ (n+1)^2 + (n+1)n + n^2 &= 3p^2q \\ 3n^2 + 3n + 2 &= 3p^2q. \end{aligned}$$

  The left-hand side is congruent to 1 modulo 3, while the right-hand side is a multiple of 3, so this is impossible.

- If $p + q - n = 3$, $p + q = n + 3$, we have

$$\begin{aligned} (p+q)^2 + (p+q)n + n^2 &= p^2q \\ (n+3)^2 + (n+3)n + n^2 &= p^2q \\ 3n^2 + 9n + 9 &= p^2q \\ 3(n^2 + 3n + 3) &= p^2q. \end{aligned}$$

  Therefore, $3 \mid p^2q$, and hence $3 \mid p$ or $3 \mid q$, and hence either p or q must be 3 and the other one is n. However, we have concluded that $p+q-n < p \iff q < n$ and $p+q-n < q \iff p < n$, which makes this impossible.

This shows that it is impossible for primes p, q and integer n such that $p^3 + q^3 + 3pq^2 = n^3$, which shows that there are no primes p, q such that $p^3 + q^3 + 3pq^2$ is the cube of an integer.

2023.3 Question 6

1. Since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, we have $e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$, and hence

$$\begin{aligned}\cosh^2 x &= \frac{\left(e^x + e^{-x}\right)^2}{4} \\ &= \frac{e^{2x} + e^{-2x} + 2}{4} \\ &= \frac{\sum_{k=0}^{\infty} \frac{(2x)^k}{k!} + \sum_{k=0}^{\infty} \frac{(-2x)^k}{k!} + 2}{4} \\ &= \frac{2\sum_{k=0}^{\infty} \frac{(2x)^{2k}}{k!} + 2}{4} \\ &\geq \frac{2 \cdot \frac{(2x)^0}{0!} + 2 \cdot \frac{(2x)^2}{2!} + 2}{4} \\ &= \frac{4 + 4x^2}{4} \\ &= 1 + x^2,\end{aligned}$$

so

$$\cosh^2 x \geq 1 + x^2.$$

Since

$$\cosh^2 x \geq 1 + x^2 > 0,$$

we have

$$0 < \frac{1}{1 + x^2} \leq \frac{1}{\cosh^2 x}.$$

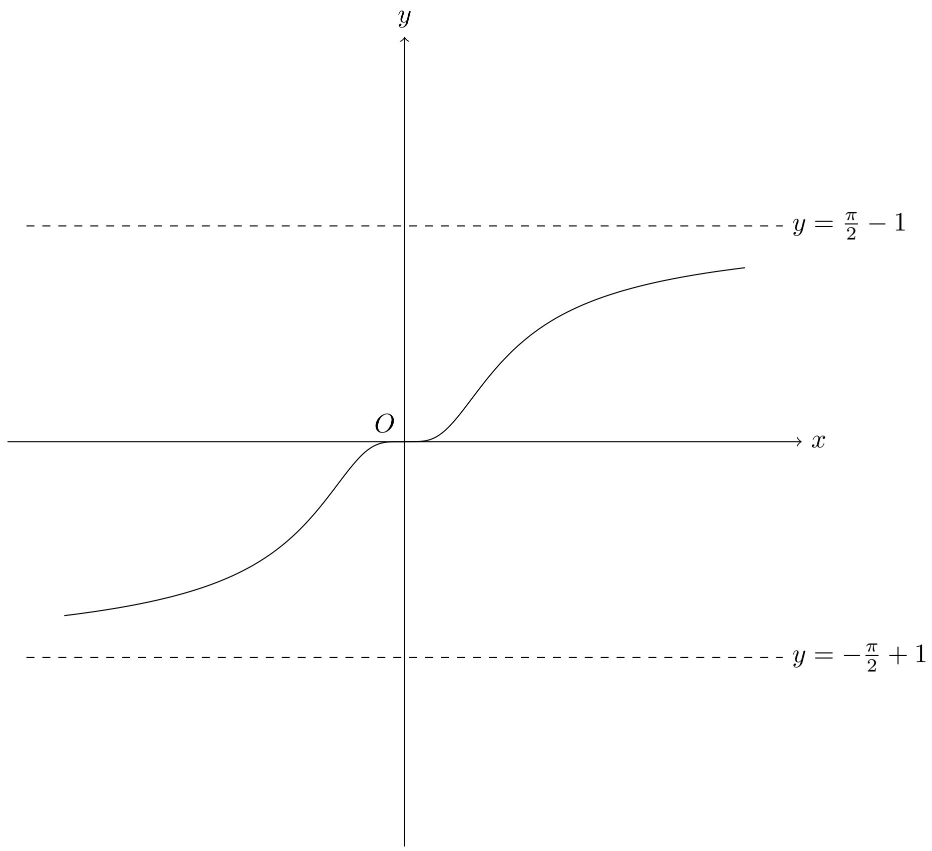
By differentiating, we have

$$\begin{aligned}f'(x) &= \frac{1}{1 + x^2} - \frac{1}{\cosh^2 x} \\ &\geq \frac{1}{1 + x^2} - \frac{1}{1 + x^2} \\ &= 0,\end{aligned}$$

which shows $f'(x) \geq 0$, meaning f is increasing.

Notice that $f'(x) = 0$ if and only if $x = 0$ (since the equal sign takes if and only if all the remaining even powers of x sum to zero, which is possible if and only if they are all zero). Also, since both functions are odd, we have $f(x) = -f(-x)$.

As $x \to \pm\infty$, $f(x) \to \pm\frac{\pi}{2} \mp 1$ respectively.



2. (a) We notice that $g(0) = \arctan 0 - \frac{1}{2}\pi \tanh 0 = 0$, and that as $x \to \infty$, $g(x) \to \frac{\pi}{2} - \frac{1}{2}\pi \cdot 1 = 0$ as well.

   Since g is not identically zero, it must be the case that it has a stationary point on $(0, \infty)$.

   Also, notice that g is odd, so the stationary points come in pairs, and there must be at least two of those.

   (b) By differentiating,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(1+x^2\right)\sinh x - x\cosh x\right] &= 2x\sinh x + \left(1+x^2\right)\cosh x - \cosh x - x\sinh x \\ &= x\sinh x + x^2\cosh x.\end{aligned}$$

   If $x \geq 0$, then $\sinh x \geq 0$ and $\cosh x \geq 0$, and hence the derivative is non-negative, meaning this function is non-decreasing.

   Therefore, for $x \geq 0$,

$$\left(1+x^2\right)\sinh x - x\cosh x \geq \left(1+0^2\right)\sinh 0 - 0 \cdot \cosh 0 = 0$$

   which shows that it is indeed non-negative, as desired.

   (c) By differentiating, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\cosh^2 x}{1+x^2} = \frac{2\cosh x\sinh x\left(1+x^2\right) - 2x\cosh^2 x}{\left(1+x^2\right)^2}.$$

   Since the denominator is always positive, showing that it is increasing for $x \geq 0$ is equivalent to showing that

$$2\cosh x\sinh x\left(1+x^2\right) - 2x\cosh^2 x = 2\cosh x\left[\sinh x(1+x^2) - x\cosh x\right]$$

   is non-negative. From the previous part, the part within the brackets is non-negative, and $\cosh x \geq 0$. Therefore, the derivative is non-negative, and this is an increasing function.

(d) By differentiating g, we have

$$\begin{aligned} g'(x) &= \frac{1}{1+x^2} - \frac{\pi}{2} \cdot \frac{1}{\cosh^2 x} \\ &= \frac{2\cosh^2 x - \pi\left(1+x^2\right)}{2\left(1+x^2\right)\cosh^2 x}. \end{aligned}$$

We first note that $g'(0) \neq 0$ since the numerator evaluates to $2 - \pi$.

Since g is odd, the curve has exactly two stationary points if and only if there is exactly one stationary point on $(0, \infty)$.

The curve has a stationary point if and only if

$$2\cosh^2 x - \pi\left(1+x^2\right) = 0,$$

if and only if

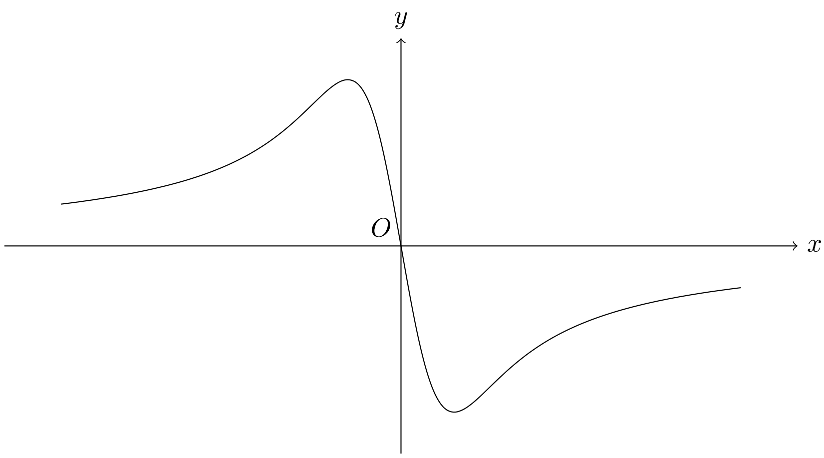
$$\frac{\cosh^2 x}{1+x^2} = \frac{\pi}{2}.$$

Since the left-hand side is increasing (and non-constant) for $x \geq 0$, there is at most one solution to this equation for $x \geq 0$.

From part (a), there is at least one stationary point for $x > 0$.

Together, this means that there is precisely one stationary point for $x > 0$, and therefore g has precisely two stationary points.

(e) The graph is as follows.



2023.3 Question 7

1. Let $u = \sqrt{x}$, we have $x = u^2$, and hence $\mathrm{d}x = 2u\,\mathrm{d}u$.

   When $x = 0, u = 0$, and when $x = 1, u = 1$.

   Therefore,

$$\begin{aligned}\int_0^1 f\left(\sqrt{x}\right)\mathrm{d}x &= \int_0^1 f(u)\cdot 2u\,\mathrm{d}u \\ &= 2\int_0^1 uf(u)\,\mathrm{d}u \\ &= 2\int_0^1 xf(x)\,\mathrm{d}x.\end{aligned}$$

2. We have

$$\begin{aligned}\int_0^1 (g(x))^2\,\mathrm{d}x &= \int_0^1 g\left(\sqrt{x}\right)\mathrm{d}x - \frac{1}{3} \\ &= 2\int_0^1 xg(x)\,\mathrm{d}x - \int_0^1 x^2\,\mathrm{d}x,\end{aligned}$$

   and hence

$$\begin{aligned}\int_0^1 \left[(g(x))^2 - 2xg(x) + x^2\right]\mathrm{d}x &= 0 \\ \int_0^1 (g(x) - x)^2\,\mathrm{d}x &= 0.\end{aligned}$$

   We have $(g(x) - x)^2 \geq 0$ for all $0 \leq x \leq 1$, and since g is continuous, we must have $g(x) - x = 0$ for all $0 \leq x \leq 1$ for the integral to evaluate to 0.

   Hence, $g(x) = x$ for $0 \leq x \leq 1$.

3. Using integration by parts, we have

$$\begin{aligned}\int_0^1 h(x)\,\mathrm{d}x &= [xh(x)]_0^1 - \int_0^1 x\,\mathrm{d}h(x) \\ &= 1\cdot h(1) - 0\cdot h(0) - \int_0^1 xh'(x)\,\mathrm{d}x \\ &= h(1) - \int_0^1 xh'(x)\,\mathrm{d}x.\end{aligned}$$

   Hence,

$$\begin{aligned}\int_0^1 (h'(x))^2\,\mathrm{d}x &= 2h(1) - 2\int_0^1 h(x)\,\mathrm{d}x - \frac{1}{3} \\ &= 2h(1) - 2\left[h(1) - \int_0^1 xh'(x)\,\mathrm{d}x\right] - \frac{1}{3} \\ &= 2\int_0^1 xh'(x)\,\mathrm{d}x - \frac{1}{3} \\ &= 2\int_0^1 xh'(x)\,\mathrm{d}x - \int_0^1 x^2\,\mathrm{d}x.\end{aligned}$$

   Therefore, similar to the previous part, we have $h'(x) = x$ for $0 \leq x \leq 1$, and hence $h(x) = \frac{x^2}{2} + h(0) = \frac{x^2}{2}$ for $0 \leq x \leq 1$.

4. First, we notice that

$$\begin{aligned}\int_0^1 e^{-ax}\,\mathrm{d}x &= -\frac{1}{a}\left[e^{-ax}\right]_0^1 \\ &= -\frac{1}{a}\cdot\left[e^{-a}-1\right] \\ &= -\frac{e^{-a}}{a}+\frac{1}{a}.\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^1 e^{ax}\left(k(x)\right)^2\mathrm{d}x &= 2\int_0^1 k(x)\,\mathrm{d}x - \int_0^1 e^{-ax}\,\mathrm{d}x + \frac{1}{a}-\frac{1}{a^2}-\frac{1}{4} \\ \int_0^1 \left[e^{ax}\left(k(x)\right)^2 - 2k(x) + e^{-ax}\right]\mathrm{d}x &= -\frac{1}{a^2}+\frac{1}{a}-\frac{1}{4} \\ \int_0^1 e^{-ax}\left[e^{2ax}\left(k(x)\right)^2 - 2e^{ax}k(x)+1\right]\mathrm{d}x &= -\left(\frac{1}{a}-\frac{1}{2}\right)^2 \\ \int_0^1 e^{-ax}\left(e^{ax}k(x)-1\right)^2\mathrm{d}x &= -\left(\frac{1}{a}-\frac{1}{2}\right)^2.\end{aligned}$$

Since $e^{-ax}>0$, and $\left(e^{ax}k(x)-1\right)^2\geq 0$, the integrand must be non-negative, and hence

$$\int_0^1 e^{-ax}\left(e^{ax}k(x)-1\right)^2\mathrm{d}x \geq 0,$$

meaning

$$\left(\frac{1}{a}-\frac{1}{2}\right)^2 \leq 0.$$

However, since this is a square, it is non-negative, and it must be the case that $\left(\frac{1}{a}-\frac{1}{2}\right)^2=0$, giving $\frac{1}{a}=\frac{1}{2}$, and hence $a=2$.
Therefore,

$$\int_0^1 e^{-ax}\left(e^{ax}k(x)-1\right)^2\mathrm{d}x = 0,$$

and since the integrand is continuous and non-negative over the interval, it must be zero everywhere for $0\leq x\leq 1$, and hence

$$e^{ax}k(x)-1=0,$$

giving

$$k(x)=e^{-ax}=e^{-2x}$$

for $0\leq x\leq 1$.

2023.3 Question 8

1. By differentiating, we have
$$f'(x) = e^{-x} - xe^{-x} = e^{-x} - f(x),$$
and
$$f''(x) = -e^{-x} - f'(x).$$
Hence,
$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y &= f''(x) + 2f'(x) + f(x) \\ &= -e^{-x} - f'(x) + f'(x) + e^{-x} - f(x) + f(x) \\ &= 0\end{aligned}$$
as desired.

   Evaluating y and y' at $x = 0$ gives us
$$y|_{x=0} = f(0) = 0 \cdot e^{-0} = 0$$
and
$$y'|_{x=0} = f'(0) = e^{-0} - f(0) = 1 - 0 = 1.$$
For the final part, we factorise $f'(x)$ to get $f'(x) = (1 - x)e^{-x}$.

   $e^{-x} > 0$ for all x. Therefore, for $x \leq 1$, $1 - x \geq 0$, and hence $f'(x) \geq 0$.

2. We let $g_1(x) = f(x) = xe^{-x}$, and we can immediately see that this differential equation is satisfied by $x \leq 1$.

   For $y = g_2(x)$ where $x \geq 1$, we notice $g_2(1) = g_1(1) = 1 \cdot e^{-1} = \frac{1}{e}$, and $g_2'(1) = g_1'(1) = f'(1) = e^{-1} - f(1) = \frac{1}{e} - \frac{1}{e} = 0$.

   If $g_2'(x) \geq 0$ for $x \geq 1$, then g_2 and g_1 satisfies the same differential equation and boundary conditions (at $x = 1$), which means they are the same solution.

   However, this is impossible since $g_1'(x) < 0$ for $x > 1$.

   Therefore, it must be the case that $g_2'(x) \leq 0$ for $x \geq 1$, and hence we have $g_2''(x) - 2g_2'(x) + g_2(x) = 0$ as our differential equation.

   The characteristic equation solves to $\lambda_{1,2} = 1$, and hence the general solution to g_2 is $g_2(x) = (A + Bx)e^x$.

   By differentiating, we have
$$g_2'(x) = Be^x + (A + Bx)e^x = Be^x + g_2(x).$$
Considering the boundary conditions, we first have $g_2(1) = \frac{1}{e}$, meaning that $(A + B)e = \frac{1}{e}$, and hence $A + B = e^{-2}$.

   We have as well $g_2'(1) = 0$, and hence $0 = B \cdot e + \frac{1}{e}$, giving us $B = -e^{-2}$.

   Therefore, $A = 2e^{-2}$, and hence
$$\begin{aligned}g_2(x) &= \left(2e^{-2} - e^{-2}x\right) e^x \\ &= e^{-2}(2 - x)e^x \\ &= (2 - x)e^{x-2}.\end{aligned}$$

3. We notice that $g_2(x) = g_1(2 - x)$, and hence $g_2(1 + x) = g_1(1 - x)$. This means they are symmetric about the line $x = 1$.

4. We first consider the range that x is in. We replace x with $c-x$ to acquire

$$\begin{aligned} r \leq c-x \leq s &\iff -r \geq -c+x \geq -s \\ &\iff -r+c \geq x \geq -s+c \\ &\iff -s+c \leq x \leq -r+c. \end{aligned}$$

In other words,

$$x \in [-s+c, -r+c] \iff c-x \in [r, s].$$

If $y = k(c-x)$, then we have $y' = (-1) \cdot k'(c-x)$, and $y'' = (-1)^2 \cdot k''(c-x) = k''(c-x)$. Therefore,

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - p\frac{\mathrm{d}y}{\mathrm{d}x} + qy &= k''(c-x) + pk'(c-x) + qk(c-x) \\ &= k''(t) + pk'(t) + qk(t) \end{aligned}$$

for $t = c-x \in [r, s]$.

Since $y = k(x)$ is a solution to the original differential equation for $r \leq x \leq s$, we must have $k''(t) + pk'(t) + qk(t) = 0$, and therefore $y = k(c-x)$ satisfies the new differential equation for $-s+c \leq x \leq -r+c$.

5. By differentiating h, we have

$$h'(x) = -e^{-x}\sin x + e^{-x}\cos x = e^{-x}(\cos x - \sin x).$$

Therefore,

$$\begin{aligned} h'\left(\frac{1}{4}\pi\right) &= e^{-\frac{1}{4}\pi}\left(\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) \\ &= e^{-\frac{1}{4}\pi}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \\ &= 0. \end{aligned}$$

Similarly,

$$h'\left(-\frac{3}{4}\pi\right) = e^{\frac{3}{4}\pi}\left(-\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right) = 0.$$

For $x \in \left[-\frac{3}{4}\pi, \frac{1}{4}\pi\right]$, the differential equation satisfied by h without the absolute value sign is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

since $h'(x) \geq 0$.

(a) Let $c = \frac{\pi}{2}$. For $x \in \left[\frac{\pi}{2} - \frac{\pi}{4}, \frac{\pi}{2} + \frac{3\pi}{4}\right] = \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$, by the previous lemma, $y = h\left(\frac{\pi}{2} - x\right)$ must be a solution to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$

Notice that

$$y' = -h'\left(\frac{\pi}{2} - x\right),$$

and that $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \iff \frac{\pi}{2} - x \in \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$, and hence $h'\left(\frac{\pi}{2} - x\right) \geq 0$, which means $y' \leq 0$. Therefore, in $x \in \left[\frac{1}{4}\pi, \frac{5}{4}\pi\right]$, $y = h\left(\frac{\pi}{2} - x\right)$ satisfies

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + 2y = 0,$$

which is the original differential equation.

We show next that this is continuously differentiable at $x = \frac{1}{4}\pi$.
It is continuous since

$$h\left(\frac{1}{4}\pi\right) = h\left(\frac{\pi}{2} - \frac{1}{4}\pi\right) = h\left(\frac{1}{4}\pi\right).$$

We have $h'(x)|_{x=\frac{1}{4}\pi} = 0$, and

$$-h'\left(\frac{\pi}{2} - x\right)\Big|_{x=\frac{1}{4}\pi} = -h'\left(\frac{\pi}{4}\right) = 0,$$

so it is continuously differentiable at $\frac{1}{4}\pi$.
Hence,

$$\begin{aligned} y &= h\left(\frac{\pi}{2} - x\right) \\ &= e^{x-\frac{\pi}{2}} \sin\left(\frac{\pi}{2} - x\right) \\ &= e^{x-\frac{\pi}{2}} \cos x, \end{aligned}$$

for $x \in \left[\frac{1}{4}\pi, \frac{5}{4}\pi\right]$.

(b) As shown above, for $x \in \left[\frac{1}{4}\pi, \frac{5}{4}\pi\right]$, $y = h\left(\frac{\pi}{2} - x\right)$ satisfies

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$

Let $c = \frac{5\pi}{2}$. For $x \in \left[\frac{5\pi}{2} - \frac{5}{4}\pi, \frac{5\pi}{2} - \frac{1}{4}\pi\right] = \left[\frac{5}{4}\pi, \frac{9}{4}\pi\right]$,

$$y = h\left(\frac{\pi}{2} - \left(\frac{5\pi}{2} - x\right)\right) = h(x - 2\pi)$$

satisfies

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$

We have

$$y' = h'(x - 2\pi) = h'\left(\frac{\pi}{2} - \left(\frac{5\pi}{2} - x\right)\right),$$

and $x \in \left[\frac{5}{4}\pi, \frac{9}{4}\pi\right] \iff \frac{5}{2}\pi - x \in \left[\frac{1}{4}\pi, \frac{5}{4}\pi\right]$, and this therefore means $h'\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) \geq 0$.
Hence, in $x \in \left[\frac{5}{4}\pi, \frac{9}{4}\pi\right]$, $y = h(x - 2\pi)$ satisfies

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + 2y = 0.$$

We show next that this is continuously differentiable at $x = \frac{5}{4}\pi$.
It is continuous since

$$h\left(\frac{\pi}{2} - \frac{5}{4}\pi\right) = h\left(-\frac{3}{4}\pi\right) = h\left(\frac{5}{4}\pi - 2\pi\right).$$

We have

$$h'\left(\frac{\pi}{2} - x\right)\Big|_{x=\frac{5}{4}\pi} = -\left. h'(x)\right|_{x=-\frac{3}{4}\pi} = -0 = 0,$$

and

$$\left. h'(x - 2\pi)\right|_{x=\frac{5}{4}\pi} = \left. h'(x)\right|_{x=-\frac{3}{4}\pi} = 0,$$

and so it is continuously differentiable at $x = \frac{5}{4}\pi$.
Therefore,

$$\begin{aligned} y &= h(x - 2\pi) \\ &= e^{-x+2\pi} \sin(x - 2\pi) \\ &= e^{2\pi - x} \sin x \end{aligned}$$

for $x \in \left[\frac{5}{4}\pi, \frac{9}{4}\pi\right]$.

2023.3 Question 11

$$\begin{aligned}\sum_{k=1}^{N} \frac{k+1}{k!} \cdot x^k &= \sum_{k=1}^{N} \frac{k}{k!} \cdot x^k + \sum_{k=1}^{N} \frac{x^k}{k!} \\ &= \sum_{k=1}^{N} \frac{1}{(k-1)!} \cdot x^k + \sum_{k=0}^{N} \frac{x^k}{k!} - \frac{x^0}{0!} \\ &= \sum_{k=0}^{N-1} \frac{1}{k!} \cdot x^{k+1} + \sum_{k=0}^{N} \frac{x^k}{k!} - 1 \\ &= x \sum_{k=0}^{N-1} \frac{x^k}{k!} + \sum_{k=0}^{N} \frac{x^k}{k!} - 1.\end{aligned}$$

We let $N \to \infty$. Using the Maclaurin Expansion for e^x, we have

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x,$$

and hence

$$\sum_{k=1}^{\infty} \frac{k+1}{k!} \cdot x^k = xe^x + e^x - 1 = (x+1)e^x - 1.$$

1. We have $Y \sim \mathrm{Po}(n)$. Let X_k be the outcome of a k-sided die, i.e. $X_k \sim \mathrm{U}(k)$. WE must have $1 \leq X_k \leq k$. The random variable D can be defined as

$$D = \begin{cases} 0, & Y = 0, \\ X_k, & Y = k. \end{cases}$$

(a)

$$\begin{aligned}\mathrm{P}(D = 0) &= \mathrm{P}(Y = 0) \\ &= e^{-n} \cdot \frac{n^0}{0!} \\ &= e^{-n}.\end{aligned}$$

(b) For $d \geq 1$, we have

$$\begin{aligned}\mathrm{P}(D = d) &= \sum_{k=d}^{\infty} \mathrm{P}(X_k = d, Y = k) \\ &= \sum_{k=d}^{\infty} \mathrm{P}(X_k = d)\,\mathrm{P}(Y = k) \\ &= \sum_{k=d}^{\infty} \frac{1}{k} \cdot e^{-n} \cdot \frac{n^k}{k!} \\ &= \sum_{k=d}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n} \right).\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{E}(D) &= \sum_{d=0}^{\infty} d\,\mathrm{P}(D = d) \\ &= \sum_{d=1}^{\infty} d\,\mathrm{P}(D = d) \\ &= \sum_{d=1}^{\infty} \left[d \sum_{k=d}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n} \right) \right].\end{aligned}$$

This summation is for

$$d \cdot \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n}\right)$$

over the set

$$\begin{aligned}(d,k) &\in \{(n,m) \mid n \geq 1, m \geq n\} \\ &= \{(n,m) \mid 1 \leq n \leq m\} \\ &= \{(n,m) \mid m \geq 1, n \leq m\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{E}(D) &= \sum_{d=1}^{\infty} \left[d \sum_{k=d}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n}\right)\right] \\ &= \sum_{(d,k)\in\{(n,m)\mid n\geq 1, m\geq n\}} d \cdot \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n}\right) \\ &= \sum_{(d,k)\in\{(n,m)\mid m\geq 1, n\leq m\}} d \cdot \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n}\right) \\ &= \sum_{k=1}^{\infty} \sum_{d=1}^{k} d \cdot \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n}\right) \\ &= \sum_{k=1}^{\infty} \left[\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n} \cdot \sum_{d=1}^{k} d\right].\end{aligned}$$

(c)

$$\begin{aligned}\mathrm{E}(D) &= \sum_{k=1}^{\infty} \left[\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n} \cdot \sum_{d=1}^{k} d\right] \\ &= \sum_{k=1}^{\infty} \left[\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n} \cdot \frac{k(k+1)}{2}\right] \\ &= \frac{e^{-n}}{2} \sum_{k=1}^{\infty} \frac{n^k(k+1)}{k!} \\ &= \frac{e^{-n}}{2} \left[(n+1) \cdot e^n - 1\right] \\ &= \frac{1}{2} \left[e^{-n} \cdot (n+1) \cdot e^n - e^{-n}\right] \\ &= \frac{1}{2} \left[(n+1) - e^{-n}\right]\end{aligned}$$

as desired.

2. $X_k \sim \mathrm{Po}(k)$ for $k = 1, 2, \cdots, n$. Let Y_n be the outcome of an n-sided die, i.e. $Y_n \sim \mathrm{U}(n)$. Therefore, $Z = X_k$ if $Y_n = k$.

(a) We have

$$\begin{aligned}
\mathrm{P}(Z=0) &= \sum_{k=1}^{n} \mathrm{P}(X_k=0, Y_n=k) \\
&= \sum_{k=1}^{n} \mathrm{P}(X_k=0)\,\mathrm{P}(Y_n=k) \\
&= \sum_{k=1}^{n} e^{-k} \cdot \frac{k^0}{0!} \cdot \frac{1}{n} \\
&= \frac{1}{n} \cdot \sum_{k=1}^{n} e^{-k} \\
&= \frac{1}{n} \cdot \frac{1-\left(e^{-1}\right)^n}{1-e^{-1}} \cdot e^{-1} \\
&= \frac{e^{-1}}{n} \cdot \frac{1-e^{-n}}{1-e^{-1}}.
\end{aligned}$$

(b) For $z \geq 1$, we have

$$\begin{aligned}
\mathrm{P}(Z=z) &= \sum_{k=1}^{n} \mathrm{P}(X_k=z, Y_n=k) \\
&= \sum_{k=1}^{n} \mathrm{P}(X_k=z)\,\mathrm{P}(Y_n=k) \\
&= \frac{1}{n} \cdot \sum_{k=1}^{n} e^{-k} \cdot \frac{k^z}{z!} \\
&= \frac{1}{nz!} \sum_{k=1}^{n} e^{-k} k^z.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{E}(Z) &= \sum_{z=0}^{\infty} z\,\mathrm{P}(Z=z) \\
&= \sum_{z=1}^{\infty} z\,\mathrm{P}(Z=z) \\
&= \sum_{z=1}^{\infty} \left[\frac{1}{n(z-1)!} \cdot \sum_{k=1}^{n} e^{-k} \cdot k^{Z}\right] \\
&= \frac{1}{n} \sum_{z=1}^{\infty} \left[\frac{1}{(z-1)!} \sum_{k=1}^{n} e^{-k} \cdot k^{z}\right] \\
&= \frac{1}{n} \sum_{z=1}^{\infty} \sum_{k=1}^{n} \left(\frac{1}{(z-1)!} \cdot e^{-k} \cdot k^{z}\right) \\
&= \frac{1}{n} \sum_{k=1}^{n} \sum_{z=1}^{\infty} \left(\frac{1}{(z-1)!} \cdot e^{-k} \cdot k^{z}\right) \\
&= \frac{1}{n} \sum_{k=1}^{n} \left[e^{-k} \cdot k \cdot \sum_{z=1}^{\infty} \frac{k^{z-1}}{(z-1)!}\right] \\
&= \frac{1}{n} \sum_{k=1}^{n} \left[e^{-k} \cdot k \cdot \sum_{z=0}^{\infty} \frac{k^{z}}{z!}\right] \\
&= \frac{1}{n} \sum_{k=1}^{n} \left[e^{-k} \cdot k \cdot e^{k}\right] \\
&= \frac{1}{n} \sum_{k=1}^{n} k \\
&= \frac{1}{n} \cdot \frac{n(n+1)}{2} \\
&= \frac{n+1}{2}.
\end{aligned}$$

Therefore, subtracting gives us

$$\begin{aligned}
\mathrm{E}(Z) - \mathrm{E}(D) &= \frac{n+1}{2} - \frac{1}{2} \cdot \left(n+1-e^{-n}\right) \\
&= \frac{1}{2} e^{-n} \\
&> 0.
\end{aligned}$$

Therefore, $\mathrm{E}(Z) > \mathrm{E}(D)$ as desired.

2023.3 Question 12

1. There are $\binom{2n}{2k}$ ways to select the socks in total.

   All $2k$ socks must be from different pairs of socks, so we have to select $2k$ pairs of socks from the n pairs available, giving us $\binom{n}{2k}$ options.

   Out of those $2k$ pairs, one of the two is selected, which gives 2^{2k}.

   Therefore, the probability is given by

$$\mathrm{P} = \frac{\binom{n}{2k} \cdot 2^{2k}}{\binom{2n}{2k}}.$$

2. There are r pairs of socks, and $2k - 2r = 2(k-r)$ socks that do not form any pairs (single).

   This gives us $\binom{n}{r}$ to select the r pairs of socks, $\binom{n-r}{2(k-r)}$ to select the $2(k-r)$ pairs from the remaining $n-r$ pairs. Finally, there is a factor of $2^{2(k-r)}$ ways to select one sock out of the $n-r$ pair.

   Hence,

$$\mathrm{P}(X_{n,k} = r) = \frac{\binom{n}{r}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n}{2k}}$$

   as desired, for $0 \leq r \leq k$.

3. By expanding out the binomial coefficients, we have

$$\begin{aligned}\mathrm{P}(X_{n,k} = r) &= \frac{\frac{n!}{(n-r)!r!} \cdot \frac{(n-r)!}{(2(k-r))!((n-r)-2(k-r))!}}{\frac{(2n)!}{(2k)!(2(n-k))!}} \cdot 2^{2(k-r)} \\ &= \frac{n!(2k)!(2(n-k))!}{(2n)!r!(2(k-r))!(n+r-2k)!} \cdot 2^{2(k-r)},\end{aligned}$$

   and hence

$$\begin{aligned}&\mathrm{P}(X_{n-1,k-1} = r-1) \\ &= \frac{(n-1)!(2(k-1))!(2((n-1)-(k-1)))!}{(2(n-1))!(r-1)!(2((k-1)-(r-1)))!((n-1)+(r-1)-2(k-1))!} \cdot 2^{2((k-1)-(r-1))} \\ &= \frac{(n-1)!(2k-2)!(2(n-k))!}{(2n-2)!(r-1)!(2(k-r))!(n+r-2k)!} \cdot 2^{2(k-r)}.\end{aligned}$$

   To show that

$$r \cdot \mathrm{P}(X_{n,k} = r) = \frac{k(2k-1)}{2n-1} \cdot \mathrm{P}(X_{n-1,k-1} = r-1),$$

   it is equivalent to showing that

$$\begin{aligned} r \cdot \frac{n!(2k)!}{(2n)!r!} &= \frac{k(2k-1)}{2n-1} \cdot \frac{(n-1)!(2k-2)!}{(2n-2)!(r-1)!} \\ r \cdot \frac{n(2k)(2k-1)}{(2n)(2n-1)r} &= \frac{k(2k-1)}{2n-1} \\ \frac{n(2k)}{2n} &= k \\ 2nk &= 2nk \end{aligned}$$

   which is true.

   Therefore, we have

$$r \cdot \mathrm{P}(X_{n,k} = r) = \frac{k(2k-1)}{2n-1} \cdot \mathrm{P}(X_{n-1,k-1} = r-1)$$

   as desired.

Therefore, the expectation can be simplified as

$$\begin{aligned}
\mathrm{E}(X_{n,k}) &= \sum_{r=0}^{k} r\,\mathrm{P}(X_{n,k} = r) \\
&= \sum_{r=1}^{k} r\,\mathrm{P}(X_{n,k} = r) \\
&= \sum_{r=1}^{k} \frac{k(2k-1)}{2n-1}\,\mathrm{P}(X_{n-1,k-1} = r-1) \\
&= \frac{k(2k-1)}{2n-1} \sum_{r=1}^{k} \mathrm{P}(X_{n-1,k-1} = r-1) \\
&= \frac{k(2k-1)}{2n-1} \sum_{r=0}^{k-1} \mathrm{P}(X_{n-1,k-1} = r-1) \\
&= \frac{k(2k-1)}{2n-1} \cdot 1 \\
&= \frac{k(2k-1)}{2n-1}
\end{aligned}$$

since $0 \leq X_{n,k} \leq k$, $0 \leq X_{n-1,k-1} \leq k-1$ and that they can only take integer values.