2023 Paper 2

2023.2 Question 1

1. If $x = \frac{1}{t}$, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{t^2}$$

and hence

$$\mathrm{d}x = -\frac{\mathrm{d}t}{t^2}.$$

Hence,

$$\begin{aligned}\int_a^b \frac{\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} &= \int_{a^{-1}}^{b^{-1}} \frac{-\,\mathrm{d}t}{t^2\left(1+\frac{1}{t^2}\right)^{\frac{3}{2}}} \\ &= \int_{a^{-1}}^{b^{-1}} \frac{-t\,\mathrm{d}t}{t^3\left(1+\frac{1}{t^2}\right)^{\frac{3}{2}}} \\ &= \int_{a^{-1}}^{b^{-1}} \frac{-t\,\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}}\end{aligned}$$

as desired.

2. We have

$$\int_{a^{-1}}^{b^{-1}} \frac{-t\,\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}} = \left[\left(1+t^2\right)^{-\frac{1}{2}}\right]_{a^{-1}}^{b^{-1}}.$$

(a)

$$\begin{aligned}\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} &= \int_{2}^{\frac{1}{2}} \frac{-t\,\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}} \\ &= \left[\left(1+t^2\right)^{-\frac{1}{2}}\right]_{2}^{\frac{1}{2}} \\ &= \left(1+\left(\frac{1}{2}\right)^2\right)^{-\frac{1}{2}} - \left(1+(2)^2\right)^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{\frac{5}{4}}} - \frac{1}{\sqrt{5}} \\ &= \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}.\end{aligned}$$

(b) Notice that the integrand is even, we have

$$\begin{aligned}
\int_{-2}^{2} \frac{\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} &= 2\int_0^2 \frac{\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} \\
&= 2\lim_{u\to 0^+} \int_u^2 \frac{\mathrm{d}x}{((1+x^2))^{\frac{3}{2}}} \\
&= 2\lim_{u\to 0^+} \int_{\frac{1}{u}}^{\frac{1}{2}} \frac{-t\,\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}} \\
&= 2\lim_{u\to \infty} \int_{u}^{\frac{1}{2}} \frac{-t\,\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}} \\
&= 2\lim_{u\to \infty} \left[\left(1+t^2\right)^{-\frac{1}{2}}\right]_u^2 \\
&= 2\cdot\left(\frac{2}{\sqrt{5}} - \lim_{u\to\infty}\frac{1}{\sqrt{1+u^2}}\right) \\
&= 2\cdot\left(\frac{2}{\sqrt{5}} - 0\right) \\
&= \frac{4}{\sqrt{5}}.
\end{aligned}$$

3. (a) Starting from the left, we have

$$\begin{aligned}
\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{(1+x^2)^2} &= \int_2^{\frac{1}{2}} \frac{-\frac{1}{t^2}\,\mathrm{d}t}{\left(1+\frac{1}{t^2}\right)^2} \\
&= \int_{\frac{1}{2}}^{2} \frac{\frac{1}{t^2}\cdot t^4\,\mathrm{d}t}{t^4\left(1+\frac{1}{t^2}\right)^2} \\
&= \int_{\frac{1}{2}}^{2} \frac{t^2\,\mathrm{d}t}{(1+t^2)^2},
\end{aligned}$$

and therefore the first equal sign is true.
As for the second equal sign, we notice that

$$\begin{aligned}
\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{(1+x^2)^2} + \int_{\frac{1}{2}}^{2} \frac{x^2\,\mathrm{d}x}{(1+x^2)^2} &= \int_{\frac{1}{2}}^{2} \frac{\left(1+x^2\right)\mathrm{d}x}{(1+x^2)^2} \\
&= \int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{1+x^2},
\end{aligned}$$

which means that

$$\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{(1+x^2)^2} = \int_{\frac{1}{2}}^{2} \frac{x^2\,\mathrm{d}x}{(1+x^2)^2} = \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{1+x^2}.$$

Hence,

$$\begin{aligned}
\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{(1+x^2)^2} &= \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{1+x^2} \\
&= \frac{1}{2}\left[\arctan x\right]_{\frac{1}{2}}^{2} \\
&= \frac{1}{2}\arctan 2 - \frac{1}{2}\arctan\frac{1}{2} \\
&= \frac{1}{2}\arctan 2 - \frac{1}{2}\left(\frac{\pi}{2} - \arctan 2\right) \\
&= \arctan 2 - \frac{\pi}{4}.
\end{aligned}$$

(b) Let $x = \frac{1}{u}$, we have $\mathrm{d}x = -\frac{1}{u^2}\,\mathrm{d}u$.

Let the integral be I, and we have

$$\begin{aligned}
I &= \int_{\frac{1}{2}}^{2} \frac{1-x}{x\left(1+x^2\right)^{\frac{1}{2}}} \, \mathrm{d}x \\
&= \int_{\frac{1}{2}}^{2} \frac{1-\frac{1}{u}}{\frac{1}{u}\left(1+\frac{1}{u^2}\right)^{\frac{1}{2}}} \cdot \frac{1}{u^2} \, \mathrm{d}u \\
&= \int_{\frac{1}{2}}^{2} \frac{u-1}{u^2\left(1+\frac{1}{u^2}\right)^{\frac{1}{2}}} \, \mathrm{d}u \\
&= \int_{\frac{1}{2}}^{2} \frac{u-1}{u\left(1+u^2\right)^{\frac{1}{2}}} \, \mathrm{d}u \\
&= -I.
\end{aligned}$$

This therefore means

$$I = \int_{\frac{1}{2}}^{2} \frac{1-x}{x\left(1+x^2\right)^{\frac{1}{2}}} \, \mathrm{d}x = 0.$$

2023.2 Question 2

1. Let
$$f(t) = \frac{2t}{1-t^2}.$$
By the double angle formula for tan, we have
$$f(\tan\theta) = \tan 2\theta.$$
Since $y = f(x)$, we have $y = f(\tan\alpha) = \tan 2\alpha$. Similarly, $z = f(y) = \tan 4\alpha$, and $x = f(z) = \tan 8\alpha$.
But since $x = x$, we must have $\tan\alpha = \tan 8\alpha$, and there must be some $k \in \mathbb{Z}$ such that
$$\alpha + k\pi = 8\alpha,$$
i.e.
$$\alpha = \frac{k\pi}{7}.$$
Since $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$ for the substitution, we have
$$\alpha = -\frac{3}{7}\pi, -\frac{2}{7}\pi, -\frac{1}{7}\pi, 0, \frac{1}{7}\pi, \frac{2}{7}\pi, \frac{3}{7}\pi,$$
and hence
$$(\alpha, 2\alpha, 4\alpha) = (0,0,0), \left(\pm\frac{1}{7}\pi, \pm\frac{2}{7}\pi, \pm\frac{4}{7}\pi\right), \left(\pm\frac{2}{7}\pi, \pm\frac{4}{7}\pi, \pm\frac{8}{7}\pi\right), \left(\pm\frac{3}{7}\pi, \pm\frac{6}{7}\pi, \pm\frac{12}{7}\pi\right),$$
which means
$$(x,y,z) = (\tan 0, \tan 0, \tan 0),$$
or
$$(x,y,z) = \left(\tan\pm\frac{1}{7}\pi, \tan\pm\frac{2}{7}\pi, \tan\pm\frac{4}{7}\pi\right) = \left(\tan\pm\frac{1}{7}\pi, \tan\pm\frac{2}{7}\pi, \tan\mp\frac{3}{7}\pi\right),$$
or
$$(x,y,z) = \left(\tan\pm\frac{2}{7}\pi, \tan\pm\frac{4}{7}\pi, \tan\pm\frac{8}{7}\pi\right) = \left(\tan\pm\frac{2}{7}\pi, \tan\mp\frac{3}{7}\pi, \tan\pm\frac{1}{7}\pi\right),$$
or
$$(x,y,z) = \left(\tan\pm\frac{3}{7}\pi, \tan\pm\frac{6}{7}\pi, \tan\pm\frac{12}{7}\pi\right) = \left(\tan\pm\frac{3}{7}\pi, \tan\mp\frac{1}{7}\pi, \tan\mp\frac{2}{7}\pi\right).$$

2. Let
$$g(t) = \frac{3t-t^3}{1-3t^2}.$$
The triple angle formula for tan is given by
$$\begin{aligned}\tan 3\theta &= \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} \\ &= \frac{\tan\theta + \frac{2\tan\theta}{1-\tan^2\theta}}{1 - \tan\theta\frac{2\tan\theta}{1-\tan^2\theta}} \qquad = \frac{\tan\theta(1-\tan^2\theta) + 2\tan\theta}{(1-\tan^2\theta) - \tan\theta(2\tan\theta)} \\ &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta},\end{aligned}$$
and hence
$$g(\tan\theta) = \tan 3\theta.$$
Let $x = \tan\alpha$ for $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. We must have $y = \tan 3\alpha$, $z = \tan 9\alpha$ and $x = \tan 27\alpha$.
There must exist some $k \in \mathbb{Z}$ such that
$$\alpha + k\pi = 27\alpha,$$

and hence
$$\alpha = \frac{k\pi}{26}.$$

It must be the case that $-13 < k < 13$, and this leads to $-12 \leq k \leq 12$. These all lead to distinct values of x.

We already have $\alpha \neq t\pi + \frac{\pi}{2}$ for any $t \in \mathbb{Z}$.

We still verify that $2\alpha \neq t\pi + \frac{\pi}{2}$. We have that

$$\begin{aligned} 2\alpha - \frac{\pi}{2} &= \frac{k\pi}{13} - \frac{\pi}{2} \\ &= \frac{(2k-13)\pi}{26}. \end{aligned}$$

$2k - 13$ cannot be a multiple of 13 apart from $k = 0$ (in which case it is still not a multiple of 26), hence not of 26, and hence $2\alpha \neq t\pi + \frac{\pi}{2}$.

A similar reasoning applies for 4α:

$$\begin{aligned} 4\alpha - \frac{\pi}{2} &= \frac{2k\pi}{13} - \frac{\pi}{2} \\ &= \frac{(4k-13)\pi}{26}. \end{aligned}$$

$4k - 13$ cannot be a multiple of 13 apart from $k = 0$ (in which case it is still not a multiple of 26), hence not of 26, and hence $4\alpha \neq t\pi + \frac{\pi}{2}$.

Therefore, all 25 values of k leads to pairs of solutions for (x, y, z), and they must all be distinct (since xs) are distinct.

Therefore, there are 25 pairs of distinct real solutions to the simultaneous solutions.

3. (a) Let $h(t) = 2t^2 - 1$. Notice that by the cosine double angle formula,
$$h(\cos\theta) = \cos 2\theta.$$
If $|x|, |y|, |z| \leq 1$, let $x = \cos\alpha$ for $0 \leq \alpha \leq \pi$. We must have $y = \cos 2\alpha, z = \cos 4\alpha$, and $x = \cos 8\alpha$, leading to $\cos\alpha = \cos 8\alpha$.
Hence, we must have, for $k \in \mathbb{Z}$, that
$$8\alpha = 2k\pi \pm \alpha,$$
which gives
$$\alpha = \frac{2k\pi}{7}$$
or
$$\alpha = \frac{2k\pi}{9}.$$
Therefore, we have
$$\alpha = 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{6\pi}{9}, \frac{8\pi}{9}$$
which gives 8 pairs of solutions for (x, y, z).

(b) We have $x = h^3(x)$, and hence x satisfies a polynomial with degree 8. Hence, there are at most 8 distinct real roots for x, and since there are 8 of them for which $|x| \leq 1$, it must be the case that they are all of them. Hence, all solutions to the equations satisfy $|x|, |y|, |z| \leq 1$.

2023.2 Question 3

1. (a) If n is odd, then p must be negative when either $x \gg 0$ or $x \ll 0$, for a sufficiently large $|x|$, since the leading term (term with x^n) will be sufficiently large at this point. Since $p(x) > 0$, n must be even. Furthermore, the leading term coefficient must be positive.

   For $0 \le k \le n$, $p^{(k)}(x)$ is an $n-k$ degree polynomial. Hence, q is also a degree n polynomial, with a positive leading term coefficient. This means when $|x|$ is sufficiently large, the leading term will be sufficiently positive and q will be positive.

   (b) We would like to show that $q(x) - q'(x) = p(x)$, and we have

$$\begin{aligned}
q(x) - q'(x) &= \sum_{k=0}^{n} p^{(k)}(x) - \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{n} p^{(k)}(x) \\
&= \sum_{k=0}^{n} p^{(k)}(x) - \sum_{k=0}^{n} p^{(k+1)}(x) \\
&= \sum_{k=0}^{n} p^{(k)}(x) - \sum_{k=1}^{n+1} p^{(k)}(x) \\
&= p^{(0)}(x) - p^{(n+1)}(x) \\
&= p(x) - 0 \\
&= p(x),
\end{aligned}$$

   as desired.

2. (a) If $q'(x) = 0$ for some x, then $0 = p(x) - q(x)$, giving $p(x) = q(x)$ for that point. This means $p(x)$ and $q(x)$ will meet at that point, proving precisely $p(x)$ and $q(x)$ meet at every stationary point of $y = q(x)$.

   This means q has all local minimums being positive, since they must be stationary points, situated on p as well, being positive.

   Since q is an even-degree polynomial, it must also be the case that one of the local minimums is a global minimum, which is positive.

   Hence, q is always positive, and $q(x) > 0$ for all x.

   (b) By differentiating, we have

$$\begin{aligned}
\frac{\mathrm{d}e^{-x}q(x)}{\mathrm{d}x} &= e^{-x}q'(x) - e^{-x}q(x) \\
&= e^{-x}(q'(x) - q(x)) \\
&= -e^{-x}p(x).
\end{aligned}$$

   We have $e^{-x} > 0$ and $p(x) > 0$ for all x, which means the gradient is always negative, which shows that $e^{-x}q(x)$ is decreasing.

   For sufficiently large x, $q(x) > 0$, and hence $e^{-x}q(x) > 0$ for sufficiently large x.

   Since this function is decreasing, we can conclude that $e^{-x}q(x) > 0$ for all x, and since e^{-x} is always positive, it must be the case that $q(x) > 0$ for all x.

   (c) Let the upper bound of the integral be N. Using integration by parts, we have

$$\begin{aligned}
\int_0^N p(x+t)e^{-t}\,\mathrm{d}t &= -\int_0^N p(x+t)\,\mathrm{d}e^{-t} \\
&= -\left[p(x+t)e^{-t}\right]_0^N + \int_0^N e^{-t}\,\mathrm{d}p(x+t) \\
&= p(x) - p(x+N)e^{-N} + \int_0^N p'(x+t)e^{-t}\,\mathrm{d}t.
\end{aligned}$$

   Let $N \to \infty$, $e^{-N}p(x+N) \to 0$ since an exponential dominates a polynomial. Hence,

$$\int_0^\infty p(x+t)e^{-t}\,\mathrm{d}t = p(x) + \int_0^\infty p^{(1)}(x+t)e^{-t}\,\mathrm{d}t$$

as desired.

Repeating this process, we have

$$\begin{aligned}
\int_0^\infty p(x+t)e^{-t}\,\mathrm{d}t &= p(x) + \int_0^\infty p^{(1)}(x+t)e^{-t}\,\mathrm{d}t \\
&= p(x) + p^{(1)}(x) + \int_0^\infty p^{(2)}(x+t)e^{-t}\,\mathrm{d}t \\
&= \cdots \\
&= p(x) + p^{(1)}(x) + \cdots + p^{(n)}(x) + \int_0^\infty p^{(n+1)}(x+t)e^{-t}\,\mathrm{d}t \\
&= \sum_{k=0}^{n} p^{(k)}(x) + \int_0^\infty 0\,\mathrm{d}t \\
&= q(x) + 0 \\
&= q(x),
\end{aligned}$$

as desired.

Since the integrand of this integral is positive for all $t \geq 0$, the integral must evaluate to a positive value, and hence $q(x) > 0$ for all x as desired.

2023.2 Question 4

1. We have

$$\begin{aligned}\left(x-\sqrt{2}\right)^2&=3\\x^2-2\sqrt{2}x+2&=3\\x^2-1&=2\sqrt{2}x\\x^4-2x^2+1&=8x^2\\x^4-10x^2+1&=0\end{aligned}$$

as desired.

If $f(x)=x^4-10x^2+1$, we notice that $x=\sqrt{2}+\sqrt{3}$ satisfies $\left(x-\sqrt{2}\right)^2=\left(\sqrt{3}\right)^2=3$, and hence $f\left(\sqrt{2}+\sqrt{3}\right)=0$ as desired.

2. We have

$$\begin{aligned}\left(x-\left(\sqrt{2}+\sqrt{3}\right)\right)^2&=\left(\sqrt{5}\right)^2=5\\x^2-2\left(\sqrt{2}+\sqrt{3}\right)x+2+3+2\sqrt{6}&=5\\x^2+2\sqrt{6}&=2\left(\sqrt{2}+\sqrt{3}\right)x\\x^4+2\cdot2\sqrt{6}\cdot x^2+\left(2\sqrt{6}\right)^2&=4\left(\sqrt{2}+\sqrt{3}\right)^2x^2\\x^4+4\sqrt{6}x^2+24&=4\left(5+2\sqrt{6}\right)x^2\\x^4+4\sqrt{6}x^2+24&=20x^2+8\sqrt{6}x^2\\x^4-20x^2+24&=4\sqrt{6}x^2\\\left(x^4-20x^2+24\right)^2&=\left(4\sqrt{6}x^2\right)^2\\x^8-40x^6+448x^4-960x^2+576&=96x^4\\x^8-40x^6+352x^4-960x^2+576&=0.\end{aligned}$$

Therefore, the polynomial

$$g(x)=x^8-40x^6+352x^4-960x^2+576$$

satisfies $g\left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right)=0$ as desired.

3. If $t=a,b,c$ are solutions to the cubic equation $t^3-3t+1=0$ in t, then $t=a+\sqrt{2},b+\sqrt{2},c+\sqrt{2}$ are solutions to the cubic equation in t

$$\begin{aligned}\left(y-\sqrt{2}\right)^3-3\left(t-\sqrt{2}\right)+1&=0\\t^3-3\sqrt{2}t^2+6t-2\sqrt{2}-3t+3\sqrt{2}+1&=0\\t^3+3t+1&=3\sqrt{2}t^2-\sqrt{2}\\t^6+6t^4+2t^3+9t^2+6t+1&=18t^4-12t^2+2\\t^6-12t^4+2t^3+21t^2+6t-1&=0.\end{aligned}$$

Therefore, the polynomial

$$h(x)=x^6-12x^4+2x^3+21x^2+6x-1$$

satisfies $h\left(a+\sqrt{2}\right)=h\left(b+\sqrt{2}\right)=h\left(c+\sqrt{2}\right)=0$ as desired.

4. We have

$$\begin{aligned}
\left(x - \sqrt[3]{2}\right)^3 &= 3 \\
x^3 - 3\sqrt[3]{2}x^2 + 3\sqrt[3]{4}x - 2 &= 3 \\
x^3 - 5 &= 3\sqrt[3]{2}x^2 - 3\sqrt[3]{4}x \\
x^3 - 5 &= 3\sqrt[3]{2}x\left(x - \sqrt[3]{2}\right) \\
x^3 - 5 &= 3\sqrt[3]{2}x \cdot \sqrt[3]{3} \\
x^3 - 5 &= 3\sqrt[3]{6}x \\
x^9 - 15x^6 + 75x^3 - 125 &= 162x^3 \\
x^9 - 15x^6 - 87x^3 - 125 &= 0.
\end{aligned}$$

Therefore, the polynomial

$$k(x) = x^9 - 15x^6 - 87x^3 - 125 = 0$$

satisfies $k\left(\sqrt[3]{2} + \sqrt[3]{3}\right) = 0$.

2023.2 Question 5

1. (a) By rearranging, we have

$$x_{n+1} = 1 + \frac{1}{x_n + 1},$$

and $x_n \geq 1$ for $n = 0$.
If $x_n \geq 1$ for some $n = k \geq 0$, we must have $\frac{1}{x_k+1} > 0$, and hence

$$x_{k+1} = 1 + \frac{1}{x_k + 1} > 1,$$

and so $x_{k+1} \geq 1$.
Hence, by the principle of mathematical induction, $x_n \geq 1$ for all $n \in \mathbb{N}$.

(b) We have

$$\begin{aligned} x_{n+1}^2 - 2 &= \left(1 + \frac{1}{x_n + 1}\right)^2 - 2 \\ &= 1 + \frac{2}{x_n + 1} + \frac{1}{(x_n + 1)^2} - 2 \\ &= \frac{1 + 2(x_n + 1) - (x_n + 1)^2}{(x_n + 1)^2} \\ &= \frac{1 + 2x_n + 2 - x_n^2 - 2x_n - 1}{(x_n + 1)^2} \\ &= \frac{-x_n^2 + 2}{(x_n + 1)^2} \\ &= -\frac{x_n^2 - 2}{(x_n + 1)^2}. \end{aligned}$$

Since

$$\frac{1}{(x_n + 1)^2} > 0,$$

it must be $x_{n+1}^2 - 2$ and $x_n^2 - 2$ must take opposite signs.

$$\begin{aligned} \left|x_{n+1}^2 - 2\right| &= \frac{1}{(x_n + 1)^2}\left|x_n^2 - 2\right| \\ &\leq \frac{1}{(1 + 1)^2}\left|x_n^2 - 2\right| \\ &= \frac{1}{4}\left|x_n^2 - 2\right|. \end{aligned}$$

(c) $x_0^2 - 2 = -1 < 0$, and so $x_n^2 - 2 < 0$ for all even n, and > 0 for all odd n.
Hence, $x_{10}^2 - 2 < 0$,and hence $x_{10}^2 \leq 2$.
We have

$$\begin{aligned} \left|x_0^2 - 2\right| &= |1 - 2| = 1, \\ \left|x_1^2 - 2\right| &\leq \frac{1}{4}\left|x_0^2 - 2\right| = \frac{1}{4}, \\ &\vdots \\ \left|x_n^2 - 2\right| &\leq \frac{1}{4^n}, \end{aligned}$$

and hence

$$\left|x_1 0^2 - 2\right| \leq \frac{1}{4^{10}} = \frac{1}{2^{20}}.$$

We have

$$\begin{aligned}2^{20} &= \left(2^{20}\right)^2 \\ &= 1024^2 \\ &> \left(10^3\right)^2 \\ &= 10^6,\end{aligned}$$

and so

$$\left|x_{10}^2 - 2\right| = 2 - x_{10}^2 < 10^{-6},$$

and hence

$$2 - 10^{-6} < x_{10}^2 < 2,$$

which gives

$$2 - 10^{-6} \leq x_{10}^2 \leq 2.$$

2. (a) We have

$$\begin{aligned}y_{n+1} - \sqrt{2} &= \frac{y_n^2 - 2\sqrt{2}y_n}{2y_n} \\ &= \frac{y_n^2 - 2\sqrt{2}y_n + \left(\sqrt{2}\right)^2}{2y_n} \\ &= \frac{\left(y_n - \sqrt{2}\right)^2}{2y_n}.\end{aligned}$$

$y_n \geq 1$ is true for the base case $n = 0$.
If it is true for $n = k$, we have

$$\frac{\left(y_n - \sqrt{2}\right)^2}{2y_n} \geq 0,$$

and so $y_{n+1} - \sqrt{2} \geq 0$, and hence $y_{n+1} \geq \sqrt{2} \geq 1$ as desired.
In fact, we can conclude that $y_n \geq \sqrt{2}$ for all $n \geq 1$.

(b) Since $y_n \geq 1$, we have $0 \leq \frac{1}{y_n} \leq 1$, and hence we have

$$y_{n+1} - \sqrt{2} \leq \frac{\left(y_n - \sqrt{2}\right)^2}{2}.$$

We aim to show the desired result by induction on n. The base case when $n = 1$ is

$$y_1 = \frac{y_0^2 + 2}{2y_0} = \frac{1^2 + 2}{2} = \frac{3}{2},$$

and

$$\text{RHS} = 2 \cdot \left(\frac{\sqrt{2} - 1}{2}\right)^{2 \cdot 1} = 2 \cdot \frac{2 + 1 - 2\sqrt{2}}{4} = \frac{3}{2} - \sqrt{2},$$

and hence

$$\text{LHS} = y_1 - \sqrt{2} = \frac{3}{2} - \sqrt{2} \leq \text{RHS}$$

as desired.

Now we assume the desired result is true for some $n = k$. For $n = k + 1$,

$$\begin{aligned} y_{k+1} - \sqrt{2} &\leq \frac{\left(y_k - \sqrt{2}\right)^2}{2} \\ &\leq \frac{\left[2 \cdot \left(\frac{\sqrt{2}-1}{2}\right)^{2^k}\right]^2}{2} \\ &= \frac{4 \cdot \left(\frac{\sqrt{2}-1}{2}\right)^{2^k \cdot 2}}{2} \\ &= 2 \cdot \left(\frac{\sqrt{2}-1}{2}\right)^{2^{k+1}}, \end{aligned}$$

which is precisely the desired statement for $n = k + 1$.

So the desired is true for the base case where $n = 1$. Given it is true for some $n = k$, it is true for $n = k + 1$. Hence, by the principle of mathematical induction,

$$y_n - \sqrt{2} \leq 2 \cdot \left(\frac{\sqrt{2}-1}{2}\right)^{2^n}$$

for all $n \geq 1$.

(c) First, we have $y_{10} \geq \sqrt{2}$ by the stronger bound found for the first part. Additionally,

$$\begin{aligned} y_{10} - \sqrt{2} &\leq 2 \cdot \left(\frac{\sqrt{2}-1}{2}\right)^{2^{10}} \\ &\leq 2 \cdot \left(\frac{\frac{1}{2}}{2}\right)^{2^{10}} \\ &= 2 \cdot \left(\frac{1}{2^2}\right)^{2^{10}} \\ &= 2 \cdot \left(\frac{1}{2}\right)^{2^{10} \cdot 2} \\ &= \frac{2}{2^{2^{11}}} \\ &= \frac{1}{2^{2^{11}-1}}. \end{aligned}$$

For the bound, notice that

$$\begin{aligned} \frac{1}{2^{2^{11}-1}} &= \frac{1}{2^{2048-1}} \\ &= \frac{1}{2^{2047}} \\ &< \frac{1}{2^{2040}} \\ &= \frac{1}{(2^{10})^{204}} \\ &< \frac{1}{(10^3)^{204}} \\ &< \frac{1}{(10^3)^{200}} \\ &= \frac{1}{10^{600}}, \end{aligned}$$

and so

$$y_{10} \leq \sqrt{2} + 10^{-600}.$$

Hence, we can conclude

$$\sqrt{2} \leq y_{10} \leq \sqrt{2} + 10^{-600}$$

as desired.

2023.2 Question 6

The base case is when $n = 1$, and we have

$$\text{LHS} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 0+1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$\text{RHS} = \mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

so LHS = RHS holds for $n = 1$.

Assume that for some $n = k \geq 1$, the original statement is true.

For $n = k + 1$, we have

$$\begin{aligned} \text{LHS} &= \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} F_k + F_{k-1} & F_{k-1} + F_{k-2} \\ F_k & F_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_k & F_{k-1} \\ F_{k-1} & F_{k-2} \end{pmatrix} \\ &= \mathbf{Q} \cdot \mathbf{Q}^k \\ &= \mathbf{Q}^{k+1} \\ &= \text{RHS}. \end{aligned}$$

So, the original statement holds for $n = 1$ base case, and assuming it holds for some $n = k \geq 1$, it holds for $n = k + 1$. Hence, by the principle of mathematical induction, for all positive integers n, we have

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \mathbf{Q}^n.$$

1. We have

$$\det \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = F_{n+1}F_{n-1} - F_n^2,$$

and on the other hand

$$\begin{aligned} \det \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} &= \det(\mathbf{Q}^n) \\ &= \det(\mathbf{Q})^n \\ &= (1 \times 0 - 1 \times 1)^n \\ &= (-1)^n. \end{aligned}$$

Hence,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

for all positive integers n.

2. On one hand,

$$\mathbf{Q}^{m+n} = \begin{pmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{pmatrix},$$

and on the other hand,

$$\begin{aligned} \mathbf{Q}^{m+n} &= \mathbf{Q}^m \cdot \mathbf{Q}^n \\ &= \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \end{aligned}$$

By comparing the top-right entry, we have $F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$ for all positive integers m and n.

3.

$$\begin{aligned}
\text{LHS} &= \mathbf{Q}^2 \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \mathbf{I} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \mathbf{I} + \mathbf{Q} \\
&= \text{RHS}
\end{aligned}$$

as desired.

(a) On one hand, we have

$$\begin{aligned}
(\mathbf{I} + \mathbf{Q})^n &= \sum_{k=0}^{n} \binom{n}{k} \mathbf{Q}^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
(\mathbf{I} + \mathbf{Q})^n &= \left(\mathbf{Q}^2\right)^n \\
&= \mathbf{Q}^{2n} \\
&= \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix}.
\end{aligned}$$

Hence, comparing the top-right entry gives us

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k$$

as desired.

(b) Notice that,

$$\begin{aligned}
\mathbf{Q}^3 &= \mathbf{Q} \cdot \mathbf{Q}^2 \\
&= \mathbf{Q} \left(\mathbf{I} + \mathbf{Q}\right) \\
&= \mathbf{Q} + \mathbf{Q}^2 \\
&= \mathbf{Q} + (\mathbf{I} + \mathbf{Q}) \\
&= \mathbf{I} + 2\mathbf{Q}.
\end{aligned}$$

Hence, on one hand, we have

$$\begin{aligned}
(\mathbf{I} + 2\mathbf{Q})^n &= \sum_{k=0}^{n} \binom{n}{k} (2\mathbf{Q})^k \\
&= \sum_{k=0}^{n} \binom{n}{k} 2^k \mathbf{Q}^k \\
&= \sum_{k=0}^{n} \binom{n}{k} 2^k \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}(\mathbf{I}+2\mathbf{Q})^n &= \left(\mathbf{Q}^3\right)^n \\ &= \mathbf{Q}^{3n} \\ &= \begin{pmatrix} F_{3n+1} & F_{3n} \\ F_{3n} & F_{3n-1} \end{pmatrix}.\end{aligned}$$

Comparing the top-right entry gives us

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} 2^k F_k.$$

Also,

$$\begin{aligned}\mathbf{Q}^{3n} &= \mathbf{Q}^n \cdot \mathbf{Q}^{2n} \\ &= \mathbf{Q}^n \sum_{k=0}^{n} \binom{n}{k} \mathbf{Q}^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \mathbf{Q}^{n+k}.\end{aligned}$$

Hence,

$$\begin{pmatrix} F_{3n+1} & F_{3n} \\ F_{3n} & F_{3n-1} \end{pmatrix} = \sum_{k=0}^{n} \binom{n}{k} \begin{pmatrix} F_{n+k+1} & F_{n+k} \\ F_{n+k} & F_{n+k-1} \end{pmatrix},$$

and comparing the top-right entry gives us

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} F_{n+k}$$

as desired.

(c) Consider $\mathbf{P} = \mathbf{I} - \mathbf{Q}$, we have

$$\mathbf{P} = \mathbf{I} - \mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} F_0 & -F_1 \\ -F_1 & F_2 \end{pmatrix}.$$

We experiment $\mathbf{P}^n$ for small ns.

$$\begin{aligned}\mathbf{P}^2 &= \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \\ \mathbf{P}^3 &= \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}, \\ \mathbf{P}^4 &= \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}.\end{aligned}$$

We claim that

$$\mathbf{P}^n = \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix}$$

and we aim to show this by induction on n.

The base case where $n = 1$ is already shown above. Assume that this statement is true for some $n = k \geq 1$, for $n = k + 1$,

$$\begin{aligned}
\text{LHS} &= \mathbf{P}^{k+1} \\
&= \mathbf{P} \cdot \mathbf{P}^k \\
&= \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} F_{k-1} & -F_k \\ -F_k & F_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} F_k & -F_{k+1} \\ -F_{k-1} - F_k & F_k + F_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} F_k & -F_{k+1} \\ -F_{k+1} & F_{k+2} \end{pmatrix} \\
&= \text{RHS}.
\end{aligned}$$

So the claim is true for the base case $n = 1$. Given it is true for some $n = k$, it is true for $n = k + 1$. Hence, by the principle of mathematical induction, this statement is true for all positive integers n.
This means, we have

$$(\mathbf{I} - \mathbf{Q})^n = \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix},$$

and hence

$$\begin{aligned}
\mathbf{Q}^n (\mathbf{I} - \mathbf{Q})^n &= \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} F_{n+1}F_{n-1} - F_n^2 & \\ & F_{n+1}F_{n-1} - F_n^2 \end{pmatrix} \\
&= (-1)^n \mathbf{I}.
\end{aligned}$$

On the other hand, using the binomial theorem, we also have

$$\begin{aligned}
\mathbf{Q}^n (\mathbf{I} - \mathbf{Q})^n &= \mathbf{Q}^n \sum_{k=0}^{n} \binom{n}{k} (-\mathbf{Q})^k \\
&= \mathbf{Q}^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mathbf{Q}^k \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mathbf{Q}^{n+k},
\end{aligned}$$

and so

$$(-1)^n \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \begin{pmatrix} F_{n+k+1} & F_{n+k} \\ F_{n+k} & F_{n+k-1} \end{pmatrix}.$$

By comparing the top-right entry, we have

$$\begin{aligned}
0 &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{n+k} \\
(-1)^n \cdot 0 &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} F_{n+k} \\
0 &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} F_{n+k}
\end{aligned}$$

as desired.

2023.2 Question 7

1. Let $z = a + ib$ and $|z| = \sqrt{a^2 + b^2}$. Let $w = c + id$ and $|w| = \sqrt{c^2 + d^2}$.
   We have $zw = (ac - bd) + (bc + ad)i$, and hence

$$\begin{aligned}|zw| &= \sqrt{(ac-bd)^2 + (bc+ad)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 - 2abcd + b^2c^2 + a^2d^2 + 2abcd} \\ &= \sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2} \\ &= \sqrt{\left(a^2+b^2\right)\left(c^2+d^2\right)} \\ &= \sqrt{a^2+b^2}\sqrt{c^2+d^2} \\ &= |z||w|\end{aligned}$$

   as desired.

2. Let $z = 2 + i$ and $w = 10 + 11i$, we have $|z| = \sqrt{5}$ and $|w| = \sqrt{221}$.
   Multiplying them gives us $zw = (2 \times 10 - 1 \times 11) + (10 \times 1 + 2 \times 11)i = 9 + 32i$.
   We have $|zw| = |z||w|$, and hence $\sqrt{9^2 + 32^2} = \sqrt{5 \times 221}$.
   This means $9^2 + 32^2 = 5 \times 221$, and hence a possible pair is $(h, k) = (9, 32)$.

3. We have $8045 = 5 \times 1609 = \left(1^2 + 2^2\right)\left(40^2 + 3^2\right)$.
   Let $z = 2 + i, w = 3 + 40i$, $zw = (2 \times 3 - 1 \times 40) + (2 \times 40 + 3 \times 1)i = -34 + 83i$.
   Since $|zw| = |z||w|$, we must have

$$\begin{aligned}34^2 + 83^2 &= \left(1^2 + 2^2\right) \times \left(40^3 + 3^2\right) \\ &= 5 \times 1609 \\ &= 8045,\end{aligned}$$

   and hence $(m, n) = (34, 83)$ is a possible pair of solution.

4. We notice that 36 is a square number, and

$$\begin{aligned}36 \times 50805 &= 6^2\left(102^2 + 201^2\right) \\ &= 6^2 \cdot 102^2 + 6^2 \cdot 201^2 \\ &= (6 \times 102)^2 + (6 \times 201)^2 \\ &= 612^2 + 1206^2.\end{aligned}$$

   Hence, $(p, q) = (612, 1206)$ is a possible pair of solution.

5. First, we observe that $1002082 = 1002001 + 81 = 1001^2 + 9^2$, and hence similar to the previous part, we have

$$\begin{aligned}25 \times 1002082 &= 5^2\left(9^2 + 1001^2\right) \\ &= (5 \times 9)^2 + (5 \times 1001)^2 \\ &= 45^2 + 5005^2,\end{aligned}$$

   and $(r, s) = (45, 5005)$ is a possible pair of solution.
   Furthermore, since $1002082 = 1001^2 + 9^2$, and $5^2 = 4^2 + 3^2$, consider $z = 3 + 4i$, $w = 1001 + 9i$, we have

$$\begin{aligned}zw &= (3 \times 1001 - 4 \times 9) + (4 \times 1001 + 3 \times 9)i \\ &= (3003 - 36) + (4004 + 27)i \\ &= 2967 + 4031i,\end{aligned}$$

   and $(r, s) = (2967, 4031)$ is a possible pair of solution since $|zw| = |z||w|$.

Similarly, $z = 4 + 3i$ and $w = 1001 + 9i$ gives

$$\begin{aligned} zw &= (4 \times 1001 - 3 \times 9) + (3 \times 1001 + 4 \times 9)i \\ &= (4004 - 27) + (3003 + 36)i \\ &= 3977 + 3039i, \end{aligned}$$

and therefore $(R, s) = (3039, 3977)$ is another possible pair of solution.

6. We have $109 = 100 + 9 = 10^2 + 3^2$, and let $z = 10 + 3i$, $w = t + ui$, we examine the linear system of equations

$$\begin{cases} 10t - 3u = 1001, \\ 3t + 10u = 6. \end{cases}$$

This solves to $t = 92$ and $u = -27$. But since $(-27)^2 = 27^2$, we must have $(t.u) = (92, 27)$ satisfies the desired equation.

2023.2 Question 8

1. Let the tetrahedron be $OABC$, and let $|OA| = a, |OB| = b, |OC| = c, |BC| = d, |AC| = e, |AB| = f$.

   This tetrahedron is isosceles, if and only if $a = d, b = e$, and $c = f$.

   The perimeter of the face OAB is $a + b + f$, of face OBC is $b + c + d$, of face OAC is $a + c + e$, and of face ABC is $d + e + f$.

   If the tetrahedron is isosceles, $a = d, b = e$ and $c = f$, then all the faces have perimeter $a + b + c$ and are equal.

   If all faces have equal perimeter, then comparing the perimeters of faces OAB, OBC and OAC, $a + f = c + d$, $b + f = c + e$, $b + d = a + e$.

   Hence, $a - d = b - e = c - f$. Let the difference be t, and $a = d + t, b = e + t, c = f + t$.

   Comparing the perimeter of face OAB and face ABC this time, we have $(d + t) + (e + t) = d + e$, which gives $t = 0$.

   Hence, $a = d$, $b = e$, $c = f$, and the tetrahedron is isosceles.

2. Applying the cosine rule in triangle OBC, we have

   $$|\mathbf{a}|^2 = |\mathbf{b}|^2 + |\mathbf{c}|^2 - 2|\mathbf{b}||\mathbf{c}| \cos \angle COB$$

   and using the dot-product formula

   $$\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}||\mathbf{c}| \cos \angle COB,$$

   rearranging gives us

   $$2\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}|^2 + |\mathbf{c}|^2 - |\mathbf{a}|^2.$$

   Similarly, we have

   $$\begin{aligned} 2\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{c}|^2, \\ 2\mathbf{a} \cdot \mathbf{c} &= |\mathbf{a}|^2 + |\mathbf{c}|^2 - |\mathbf{b}|^2. \end{aligned}$$

   Summing these two, we get

   $$\begin{aligned} 2\mathbf{a} \cdot \mathbf{b} + 2\mathbf{a} \cdot \mathbf{c} &= 2|\mathbf{a}|^2 \\ \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} &= |\mathbf{a}|^2 \\ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= |\mathbf{a}|^2. \end{aligned}$$

3. Let $\mathbf{g}$ be the position vector for G. $|OG| = |\mathbf{g}| = \frac{1}{4}|\mathbf{a} + \mathbf{b} + \mathbf{c}|$.

   Consider the distance between A and G.

   $$\begin{aligned} |AG| &= \left|\overrightarrow{AG}\right| \\ &= |\mathbf{g} - \mathbf{a}| \\ &= \frac{1}{4}|-3\mathbf{a} + \mathbf{b} + \mathbf{c}|. \end{aligned}$$

   We want to show that $|\mathbf{a} + \mathbf{b} + \mathbf{c}| = |-3\mathbf{a} + \mathbf{b} + \mathbf{c}|$. The following are equivalent

   $$\begin{aligned} |\mathbf{a} + \mathbf{b} + \mathbf{c}| &= |-3\mathbf{a} + \mathbf{b} + \mathbf{c}| \\ |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 &= |-3\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 \\ |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2\mathbf{a} \cdot \mathbf{b} + 2\mathbf{a} \cdot \mathbf{c} + 2\mathbf{b} \cdot \mathbf{c} &= 9|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 - 6\mathbf{a} \cdot \mathbf{b} - 6\mathbf{a} \cdot \mathbf{c} + 2\mathbf{b} \cdot \mathbf{c} \\ 8\mathbf{a} \cdot \mathbf{b} + 8\mathbf{a} \cdot \mathbf{c} &= 8|\mathbf{a}|^2 \\ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= |\mathbf{a}|^2 \end{aligned}$$

   and this is true from the previous part.

   Hence, $|OG| = |AG|$. By symmetry, $|OG| = |AG| = |BG| = |CG|$ and hence G is equidistant from all four vertices of the tetrahedron.

4. Notice that

$$\begin{aligned}|\mathbf{a}-\mathbf{b}-\mathbf{c}|^2 &= |\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2-2\mathbf{a}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{c}+2\mathbf{b}\cdot\mathbf{c} \\ &= |\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2-2|\mathbf{a}|^2+\left(|\mathbf{b}|^2+|\mathbf{c}|^2-|\mathbf{a}|^2\right) \\ &= -2|\mathbf{a}|^2+2|\mathbf{b}|^2+2|\mathbf{c}|^2 \\ &= 2\left(|\mathbf{b}|^2+|\mathbf{c}|^2-|\mathbf{a}|^2\right) \\ &= 4\mathbf{b}\cdot\mathbf{c},\end{aligned}$$

and since the left-hand side is a square, it is non-negative, which means the dot product is non-negative.

Hence, $\cos\angle BOC \geq 0$, which means it must not be obtuse. By symmetry, this means none of the angles are obtuse.

If one of them is a right angle, say $\angle BOC$, then the dot product evaluates to 0, which must mean $|\mathbf{a}-\mathbf{b}-\mathbf{c}|=0$.

Hence, $\mathbf{a}=\mathbf{b}+\mathbf{c}$, which means A lies in the plane containing O, B, C. This will not be a tetrahedron, and hence no angles can be right angles.

2023.2 Question 11

1. For some $1 \le i \le n$, we have

$$\begin{aligned}
\mathrm{P}(Y = x_i) &= \mathrm{P}(Y = X_i, Y = X_1) + \mathrm{P}(Y = X_i, Y = X_2) \\
&= \mathrm{P}(Y = x_i \mid Y = X_1) \cdot \mathrm{P}(Y = X_1) + \mathrm{P}(Y = x_i \mid Y = X_2) \cdot \mathrm{P}(Y = X_2) \\
&= \mathrm{P}(X_1 = x_i) \cdot \mathrm{P}(Y = X_1) + \mathrm{P}(X_2 = X_i) \cdot \mathrm{P}(Y = x_2) \\
&= pa_i + qb_i.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{E}(Y) &= \sum_{i=1}^{n} x_i \, \mathrm{P}(Y = x_i) \\
&= \sum_{i=1}^{n} x_i (pa_i + qb_i) \\
&= p \sum_{i=1}^{n} x_i a_i + q \sum_{i=1}^{n} x_i b_i \\
&= p \, \mathrm{E}(X_1) + q \, \mathrm{E}(X_2) \\
&= p\mu_1 + q\mu_2.
\end{aligned}$$

For the variance, we have

$$\begin{aligned}
\mathrm{E}(Y^2) &= \sum_{i=1}^{n} x_i^2 \, \mathrm{P}(Y = x_i) \\
&= \sum_{i=1}^{n} x_i^2 (pa_i + qb_i) \\
&= p \sum_{i=1}^{n} x_i^2 a_i + q \sum_{i=1}^{n} x_i^2 b_i \\
&= p \, \mathrm{E}(X_1^2) + q \, \mathrm{E}(X_2^2) \\
&= p \left(\mathrm{E}(X_1)^2 + \mathrm{Var}(X_1) \right) + q \left(\mathrm{E}(X_2)^2 + \mathrm{Var}(X_2) \right) \\
&= p \left(\mu_1^2 + \sigma_1^2 \right) + q \left(\mu_2^2 + \sigma_2^2 \right),
\end{aligned}$$

and hence

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{E}(Y^2) - \mathrm{E}(Y)^2 \\
&= p \left(\mu_1^2 + \sigma_1^2 \right) + q \left(\mu_2^2 + \sigma_2^2 \right) - (p\mu_1 + q\mu_2)^2 \\
&= p\sigma_1^2 + q\sigma_2^2 + p\mu_1^2 + q\mu_2^2 - p^2\mu_1^2 - q^2\mu_2^2 - 2pq\mu_1\mu_2 \\
&= p\sigma_1^2 + q\sigma_2^2 + p(1-p)\mu_1^2 + q(1-q)\mu_2^2 - 2pq\mu_1\mu_2 \\
&= p\sigma_1^2 + q\sigma_2^2 + pq\mu_1^2 + pq\mu_2^2 - 2pq\mu_1\mu_2 \\
&= p\sigma_1^2 + q\sigma_2^2 + pq \left(\mu_1 - \mu_2 \right)^2,
\end{aligned}$$

as desired.

2. We have

$$\mathrm{P}(B = 1) = \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{5}{6} = \frac{1}{2}.$$

Z_1 is the sum of n independent values of B, and counts the number of times when $B = 1$. Hence, $Z_1 \sim \mathrm{B}\left(n, \frac{1}{2}\right)$.
Since $n \gg 1$, we have

$$Z_1 \sim \mathrm{B}\left(n, \frac{1}{2}\right) \mathrel{\dot\sim} \mathrm{N}\left(\frac{n}{2}, \frac{n}{4}\right).$$

The probability of Z_1 being within 10 percent of its mean is given by

$$\begin{aligned}\mathrm{P}\left(\frac{n}{2}-\frac{n}{20}\leq Z_1\leq\frac{n}{2}+\frac{n}{20}\right)&=\mathrm{P}\left(-\frac{\frac{n}{20}}{\frac{\sqrt{n}}{2}}\leq Z\leq\frac{\frac{n}{20}}{\frac{\sqrt{n}}{2}}\right)\\&=\mathrm{P}\left(-\frac{\sqrt{n}}{20}\leq Z\leq\frac{\sqrt{n}}{20}\right)\end{aligned}$$

where $Z\sim\mathrm{N}(0,1)$ is the standard normal.

As $n\to\infty$, $-\frac{\sqrt{n}}{20}\to-\infty$, and $\frac{\sqrt{n}}{20}\to\infty$, and so the probability approaches $\mathrm{P}(-\infty<Z<\infty)$ which is 1.

3. Let $X_1\sim\mathrm{B}\left(n,\frac{1}{6}\right)$, and $X_2\sim\mathrm{B}\left(n,\frac{5}{6}\right)$. Z_2 has $\frac{1}{2}$ chance of taking X_1 and $\frac{1}{2}$ chance of taking X_2.
We have $\mu_1=\frac{n}{6},\mu_2=\frac{5n}{6},\sigma_1^2=\sigma_2^2=\frac{5n}{36}$.
Hence,

$$\mathrm{E}(Z_2)=\frac{1}{2}\cdot\frac{n}{6}+\frac{1}{2}\cdot\frac{5n}{6}=\frac{n}{2},$$

and

$$\mathrm{Var}(Z_2)=\frac{1}{2}\cdot\frac{5n}{36}+\frac{1}{2}\cdot\frac{5n}{36}+\frac{1}{4}\left(\frac{n}{6}-\frac{5n}{6}\right)^2=\frac{n^2}{9}+\frac{5n}{36}.$$

A normal approximation will not be a good approximation since in this case, Z_2 is bimodal – it is likely to take values close to $\frac{n}{6}$ or $\frac{5n}{6}$, but not near the mean $\frac{n}{2}$.
The bounds within 10 percent of the mean is $\frac{n}{2}\pm\frac{n}{20}$. We have

$$\begin{aligned}\mathrm{P}\left(\frac{n}{2}-\frac{n}{20}\leq Z_2\leq\frac{n}{2}+\frac{n}{20}\right)&=\frac{1}{2}\mathrm{P}\left(\frac{n}{2}-\frac{n}{20}\leq X_1\leq\frac{n}{2}+\frac{n}{20}\right)+\frac{1}{2}\mathrm{P}\left(\frac{n}{2}-\frac{n}{20}\leq X_2\leq\frac{n}{2}+\frac{n}{20}\right)\\&=\frac{1}{2}\mathrm{P}\left(\frac{n}{2}-\frac{n}{20}\leq X_1\right)+\frac{1}{2}\mathrm{P}\left(X_2\leq\frac{n}{2}+\frac{n}{20}\right)\\&=\mathrm{P}\left(\frac{n}{2}-\frac{n}{20}\leq X_1\right).\end{aligned}$$

Since n is large, we have $X_1\sim\mathrm{B}\left(n,\frac{1}{6}\right)\dot{\sim}\,\mathrm{N}\left(\frac{n}{6},\frac{5n}{36}\right)$, and hence

$$\begin{aligned}\mathrm{P}\left(\frac{n}{2}-\frac{n}{20}\leq X_1\right)&=\mathrm{P}\left(Z\geq\frac{\frac{n}{2}-\frac{n}{20}-\frac{n}{6}}{\frac{\sqrt{5n}}{6}}\right)\\&=\mathrm{P}\left(Z\geq\frac{30n-3n-10n}{10\sqrt{5n}}\right)\\&=\mathrm{P}\left(Z\geq\frac{17\sqrt{n}}{10\sqrt{5}}\right),\end{aligned}$$

and as $n\to\infty$, $\frac{17\sqrt{n}}{10\sqrt{5}}\to\infty$, and hence the probability tends to 0, as desired.

2023.2 Question 12

1. We first consider the event $Y \leq t$.

$$\begin{aligned} Y \leq t &\iff \max\{X_1, X_2, \ldots, X_n\} \leq t \\ &\iff X_1, X_2, \ldots, X_n \leq t \\ &\iff X_1 \leq t, X_2 \leq t, \cdots, X_n \leq t. \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{P}(Y \leq t) &= \mathrm{P}(X_1 \leq t, X_2 \leq t, \cdots, X_n \leq t) \\ &= \mathrm{P}(X_1 \leq t)\,\mathrm{P}(X_2 \leq t) \cdot \mathrm{P}(X_n \leq t) \\ &= [\mathrm{P}(X_1 \leq t)]^n \end{aligned}$$

as desired.

We first find the cumulative distribution function of X, F. For $0 \leq x \leq \pi$,

$$\begin{aligned} F(x) &= \int_0^x f(t)\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^x \sin t\,\mathrm{d}t \\ &= -\frac{1}{2}[\cos t]_0^x \\ &= \frac{1}{2}(1-\cos x). \end{aligned}$$

Now, let G be the cumulative distribution function of Y. We have $0 \leq Y \leq \pi$. For $0 \leq y \leq \pi$,

$$\begin{aligned} G(y) &= \mathrm{P}(Y \leq y) \\ &= [\mathrm{P}(X_1 \leq y)]^n \\ &= [F(y)]^n \\ &= \left[\frac{1}{2}(1-\cos y)\right]^n &&= \frac{1}{2^n}(1-\cos y)^n. \end{aligned}$$

Hence, the probability density function of Y, g, is given by

$$\begin{aligned} g(y) &= G'(y) \\ &= \frac{1}{2^n} \cdot n \cdot \sin x \cdot (1-\cos X)^{n-1} \\ &= \frac{n \sin x\,(1-\cos X)^{n-1}}{2^n} \end{aligned}$$

for $0 \leq t \leq \pi$, and 0 otherwise.

2. $m(n)$ is such that

$$\begin{aligned} G(m(n)) &= \frac{1}{2} \\ \frac{1}{2^n}(1-\cos m(n))^n &= \frac{1}{2} \\ (1-\cos m(n))^n &= 2^{n-1} \\ 1-\cos m(n) &= 2^{\frac{n-1}{n}} \\ \cos m(n) &= 1-2^{1-\frac{1}{n}} \\ m(n) &= \arccos\left(1-2^{1-\frac{1}{n}}\right). \end{aligned}$$

As n increases, $\frac{1}{n}$ decreases, $1-\frac{1}{n}$ increases, $2^{1-\frac{1}{n}}$ increases, $1-2^{1-\frac{1}{n}}$ increases, and so $m(n)$ increases. $m(n) \to \pi$ as $n \to \infty$.

3. By definition, we have

$$\begin{aligned}\mu(n) &= \mathrm{E}(Y)\\ &= \int_0^\pi \frac{n}{2^n} x\sin x\,(1-\cos x)^{n-1}\,\mathrm{d}x\\ &= \frac{1}{2^n}\int_0^\pi x\cdot n\sin x\,(1-\cos x)^{n-1}\,\mathrm{d}x\\ &= \frac{1}{2^n}\int_0^\pi x\cdot(1-\cos x)^n\,\mathrm{d}x\\ &= \frac{1}{2^n}\left[x\,(1-\cos x)^n\right]_0^\pi - \frac{1}{2^n}\int_0^\pi (1-\cos x)^n\,\mathrm{d}x\\ &= \frac{1}{2^n}\left[\pi\cdot(1+1)^n - 0\cdot(1-1)^n\right] - \frac{1}{2^n}\int_0^\pi (1-\cos x)^n\,\mathrm{d}x\\ &= \frac{1}{2^n}\cdot\pi\cdot 2^n - \frac{1}{2^n}\int_0^\pi (1-\cos x)^n\,\mathrm{d}x\\ &= \pi - \frac{1}{2^n}\int_0^\pi (1-\cos x)^n\,\mathrm{d}x.\end{aligned}$$

(a) By taking difference of two consecutive terms of $\mu(n)$, we have

$$\begin{aligned}\mu(n+1)-\mu(n) &= \left[\pi - \frac{1}{2^{n+1}}\int_0^\pi (1-\cos x)^{n+1}\,\mathrm{d}x\right] - \left[\pi - \frac{1}{2}\int_0^\pi (1-\cos x)^n\,\mathrm{d}x\right]\\ &= \frac{1}{2^n}\int_0^\pi (1-\cos x)^n\,\mathrm{d}x - \frac{1}{2^{n+1}}\int_0^\pi (1-\cos x)^{n+1}\,\mathrm{d}x\\ &= \frac{1}{2^{n+1}}\int_0^\pi \left[2\,(1-\cos x)^n - (1-\cos x)^{n+1}\right]\mathrm{d}x\\ &= \frac{1}{2^{n+1}}\int_0^\pi (1-\cos x)^n\,[2-(1-\cos x)]\,\mathrm{d}x\\ &= \frac{1}{2^{n+1}}\int_0^\pi (1-\cos x)^n\,(1+\cos x)\,\mathrm{d}x.\end{aligned}$$

For $0 < x < \pi$, we have $0 < \cos x < 1$, and so the integrand is positive on the interval. Hence, $\mu(n+1)-\mu(n) > 0$, and $\mu(n+1) > \mu(n)$, and hence $\mu(n)$ increases with n.

(b) On one hand, we have

$$m(2) = \arccos\left(1 - 2^{1-\frac{1}{2}}\right) = \arccos\left(1-\sqrt{2}\right).$$

On the other hand,

$$\begin{aligned}\mu(2) &= \pi - \frac{1}{4}\int_0^\pi (1-\cos x)^2\,\mathrm{d}x\\ &= \pi - \frac{1}{4}\int_0^\pi \left(1-2\cos x+\cos^2 x\right)\mathrm{d}x\\ &= \pi - \frac{1}{4}\int_0^\pi \left(1-2\cos x+\frac{\cos 2x+1}{2}\right)\mathrm{d}x\\ &= \pi - \frac{1}{4}\int_0^\pi \left(\frac{3}{2}-2\cos x+\frac{1}{2}\cos 2x\right)\mathrm{d}x\\ &= \pi - \frac{1}{4}\left(\frac{3}{2}x - 2\sin x + \frac{1}{4}\sin 2x\right)_0^\pi\\ &= \pi - \frac{1}{4}\left[\frac{3}{2}(\pi - x) - 2\,(\sin\pi - \sin 0) + \frac{1}{4}(\sin 2\pi - \sin 0)\right]\\ &= \pi - \frac{1}{4}\cdot\frac{3}{2}\pi\\ &= \frac{5}{8}\pi.\end{aligned}$$

We want to show that

$$\left(0 < \frac{1}{2}\pi <\right) \frac{5}{8}\pi < \arccos\left(1 - \sqrt{2}\right) (< \pi),$$

and this is equivalent to showing that

$$\cos \frac{5}{8}\pi > 1 - \sqrt{2}.$$

We first notice that $\cos \frac{5}{8}\pi = \cos\left(\frac{1}{2}\pi + \frac{1}{8}\pi\right)$, and notice that $\cos\left(\frac{1}{8}\pi\right)$ is such that

$$2\cos^2\left(\frac{1}{8}\pi\right) - 1 = \cos\left(2 \cdot \frac{1}{8}\pi\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

and hence

$$2\cos^2\frac{\pi}{8} = 1 + \frac{1}{\sqrt{2}} = \frac{2+\sqrt{2}}{2},$$

meaning

$$\cos\frac{\pi}{8} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

Therefore,

$$\sin^2\frac{\pi}{8} = 1 - \frac{2+\sqrt{2}}{4} = \frac{2-\sqrt{2}}{4}$$

and hence

$$\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Hence,

$$\begin{aligned}\cos\frac{5}{8}\pi &= \cos\left(\frac{1}{2}\pi + \frac{1}{8}\pi\right)\\ &= \cos\frac{1}{2}\pi\cos\frac{1}{8}\pi - \sin\frac{1}{2}\pi\sin\frac{1}{8}\pi\\ &= 0 - \sin\frac{1}{8}\pi\\ &= -\frac{\sqrt{2-\sqrt{2}}}{2}.\end{aligned}$$

Finally, we have the following being equivalent:

$$\begin{aligned}\cos\frac{5}{8}\pi &> 1 - \sqrt{2}\\ (0 >) -\frac{\sqrt{2-\sqrt{2}}}{2} &> 1 - \sqrt{2}\\ \sqrt{2} - 1 &> \frac{\sqrt{2-\sqrt{2}}}{2}\\ 2 + 1 - 2\sqrt{2} &> \frac{2-\sqrt{2}}{4}\\ 12 - 8\sqrt{2} &> 2 - \sqrt{2}\\ 7\sqrt{2} &< 10\\ 49 \cdot 2 = 98 &< 100\end{aligned}$$

is true, and hence $\mu(2) < m(2)$ as desired.