2021 Paper 3

2021.3 Question 1

1. By using the chain rule, we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t}\\ &= \frac{12\cos t - 12\sin^2 t\cos t}{12\cos^2 t\sin t}\\ &= \frac{\cos t - \sin^2 t\cos t}{\cos^2 t\sin t}\\ &= \frac{1-\sin^2 t}{\cos t\sin t}\\ &= \frac{\cos^2 t}{\cos t\sin t}\\ &= \frac{\cos t}{\sin t}\\ &= \cot t.\end{aligned}$$

Hence, at $t = \varphi$, the normal of this curve has gradient $-\tan\varphi$, and hence it has equation

$$\begin{aligned} y - \left(12\sin\varphi - 4\sin^3\varphi\right) &= -\tan\varphi\left(x - \left(-4\cos^3\varphi\right)\right)\\ y - 12\sin\varphi + 4\sin^3\varphi &= -\tan\varphi x - 4\cos^3\varphi\tan\varphi\\ \cos\varphi y - 12\sin\varphi\cos\varphi + 4\sin^3\varphi\cos\varphi &= -\sin\varphi x - 4\cos^3\varphi\sin\varphi\\ \sin\varphi x + \cos\varphi y &= 12\sin\varphi\cos\varphi - 4\sin^3\varphi\cos\varphi - 4\cos^3\varphi\sin\varphi\\ \sin\varphi x + \cos\varphi y &= 4\sin\varphi\cos\varphi\left(3 - \sin^2\varphi - \cos^2\varphi\right)\\ \sin\varphi x + \cos\varphi y &= 8\sin\varphi\cos\varphi.\end{aligned}$$

The curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ can be parametrised as $x = 8\cos^3 t$ and $y = 8\sin^3 t$:

$$\begin{aligned} x^{\frac{2}{3}} + y^{\frac{2}{3}} &= \left(8\cos^3 t\right)^{\frac{2}{3}} + \left(8\sin^3 t\right)^{\frac{2}{3}}\\ &= 4\cos^2 t + 4\sin^2 t\\ &= 4.\end{aligned}$$

Hence, the gradient of the tangent at a point is

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t}\\ &= \frac{24\sin^2 t\cos t}{-24\cos^2 t\sin t}\\ &= -\tan t,\end{aligned}$$

and the equation of the tangent at the point $t = \varphi$ is

$$\begin{aligned} y - 8\sin^3\varphi &= -\tan\varphi\left(x - 8\cos^3\varphi\right)\\ \cos\varphi y - 8\sin^3\varphi\cos\varphi &= -\sin\varphi x + 8\cos^3\varphi\sin\varphi\\ \sin\varphi x + \cos\varphi y &= 8\sin\varphi\cos\varphi\left(\sin^2\varphi + \cos^2\varphi\right)\\ \sin\varphi x + \cos\varphi y &= 8\sin\varphi\cos\varphi,\end{aligned}$$

which shows the normal to the original curve is the tangent to this new curve at $(8\cos^3\varphi, 8\sin^3\varphi)$.

2. By using the chain rule, we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t}\\ &= \frac{\cos t - \cos t + t\sin t}{-\sin t + \sin t + t\cos t}\\ &= \frac{t\sin t}{t\cos t}\\ &= \tan t.\end{aligned}$$

Hence, at $t = \varphi$, the normal of this curve has gradient $-\cot\varphi$, and hence it has equation

$$\begin{aligned}
y - (\sin\varphi - \varphi\cos\varphi) &= -\cot\varphi\,(x - (\cos\varphi + \varphi\sin\varphi)) \\
\sin\varphi y - \sin^2\varphi + \varphi\sin\varphi\cos\varphi &= -\cos\varphi x + \cos^2\varphi + \varphi\sin\varphi\cos\varphi \\
\cos\varphi x + \sin\varphi y &= \sin^2\varphi + \cos^2\varphi \\
\cos\varphi x + \sin\varphi y &= 1.
\end{aligned}$$

The distance of this normal to the origin is

$$\frac{|\cos\varphi \cdot 0 + \sin\varphi \cdot 0 - 1|}{\sqrt{\cos^2\varphi + \sin^2\varphi}} = 1,$$

which is a constant, and hence this curve is tangent to the unit circle $x^2 + y^2 = 1$.

2021.3 Question 2

1. For the first row/component in $\hat{\mathbf{\imath}}$,

$$\begin{aligned}\begin{pmatrix}1 & -x & x\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} &= 1\cdot a + (-x)\cdot b + x\cdot c\\ &= a + \frac{-ab}{b-c} = \frac{ac}{b-c}\\ &= a + \frac{ac-ab}{b-c}\\ &= a + (-a)\\ &= 0,\end{aligned}$$

and this is similar for the remaining row and components. Hence, we have

$$\begin{pmatrix}1 & -x & x\\ y & 1 & -y\\ -z & z & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

as desired.

If the matrix

$$\begin{pmatrix}1 & -x & x\\ y & 1 & -y\\ -z & z & 1\end{pmatrix}$$

is invertible, then we must have

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}1 & -x & x\\ y & 1 & -y\\ -z & z & 1\end{pmatrix}^{-1}\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

which is impossible, since a, b and c are distinct.

Hence, this matrix is not invertible, and it must have a zero-determinant, meaning

$$\begin{aligned}0 &= \det\begin{pmatrix}1 & -x & x\\ y & 1 & -y\\ -z & z & 1\end{pmatrix}\\ &= 1\cdot 1\cdot 1 + (-x)\cdot(-y)\cdot(-z) + x\cdot y\cdot z - 1\cdot(-y)\cdot z - (-x)\cdot y\cdot 1 - x\cdot 1\cdot(-z)\\ &= 1 - xyz + xyz + yz + xy + xz\\ &= xy + yz + zx + 1,\end{aligned}$$

and hence

$$xy + yz + zx = -1.$$

Since $(x+y+z)^2 \geq 0$, we have

$$\begin{aligned}0 &\leq (x+y+z)^2\\ &= x^2 + y^2 + z^2 + 2(xy+yz+zx)\\ &= \frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} + 2\cdot(-1),\end{aligned}$$

and hence

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} \geq 2,$$

as desired.

2. Consider the matrix
$$\begin{pmatrix} -2 & x & x \\ y & -2 & y \\ z & z & -2, \end{pmatrix}$$
and for the first row/component in $\hat{\mathbf{\imath}}$,
$$\begin{aligned} \begin{pmatrix} -2 & x & x \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= (-2)a + bx + cx \\ &= (-2)a + (b+c)x \\ &= (-2)a + 2a \\ &= 0, \end{aligned}$$
and similarly in the second and third rows/components, and hence
$$\begin{pmatrix} -2 & x & x \\ y & -2 & y \\ z & z & -2, \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
By similar argument as before, this matrix must have a zero determinant as well, and hence
$$\begin{aligned} 0 &= \det \begin{pmatrix} -2 & x & x \\ y & -2 & y \\ z & z & -2, \end{pmatrix} \\ &= (-2)(-2)(-2) + xyz + xyz - (-2)yz - xy(-2) - x(-2)z \\ &= -8 + 2xyz + 2xy + 2yz + 2zx, \end{aligned}$$
and hence
$$xyz + xy + yz + zx = 4,$$
as desired.

Hence, consider
$$(x+1)(y+1)(z+1) = xyz + xy + yz + zx + x + y + z + 1 = 5 + x + y + z.$$
Since a, b, c are all positive real numbers, x, y, z are as well, and hence $x + y + z > 0$, giving
$$(x+1)(y+1)(z+1) > 5,$$
which means
$$\frac{2a+b+c}{b+c} \cdot \frac{a+2b+c}{a+c} \cdot \frac{a+b+2c}{a+b} > 5,$$
and hence
$$(2a+b+c)(a+2b+c)(a+b+2c) > 5(b+c)(c+a)(a+b)$$
as desired.

Furthermore, notice that
$$\begin{aligned} x + y + z &= \frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b} \\ &> \frac{2a}{a+b+c} + \frac{2b}{a+b+c} + \frac{2c}{a+b+c} \\ &= \frac{2(a+b+c)}{a+b+c} \\ &= 2. \end{aligned}$$
Hence,
$$(x+1)(y+1)(z+1) > 7,$$

which means

$$\frac{2a+b+c}{b+c} \cdot \frac{a+2b+c}{a+c} \cdot \frac{a+b+2c}{a+b} > 7,$$

and hence

$$(2a+b+c)(a+2b+c)(a+b+2c) > 7(b+c)(c+a)(a+b)$$

as desired.

2021.3 Question 3

1. Notice that

$$\begin{aligned}
\text{LHS} &= \frac{1}{2}\left(I_{n+1}+I_{n-1}\right)\\
&= \frac{1}{2}\left(\int_0^\beta (\sec x+\tan x)^{n+1}\,\mathrm{d}x+\int_0^\beta (\sec x+\tan x)^{n-1}\,\mathrm{d}x\right)\\
&= \frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n-1}\left[(\sec x+\tan x)^2+1\right]\mathrm{d}x\\
&= \frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n-1}\left(\sec^2 x+\tan^2 x+2\sec x\tan x+1\right)\mathrm{d}x\\
&= \frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n-1}\cdot 2\left(\sec^2 x+\sec x\tan x\right)\mathrm{d}x\\
&= \int_0^\beta (\sec x+\tan x)^{n-1}\,\mathrm{d}(\sec x+\tan x)\\
&= \frac{1}{n}\left[(\sec x+\tan x)^n\right]_0^\beta\\
&= \frac{1}{n}\left((\sec\beta+\tan\beta)^n-(\sec 0+\tan 0)^n\right)\\
&= \frac{1}{n}\left((\sec\beta+\tan\beta)^n-1\right)\\
&= \text{RHS},
\end{aligned}$$

as desired.

To show the final part, we would like to show that

$$I_n < \frac{1}{2}\left(I_{n+1}+I_{n-1}\right)=\frac{1}{n}\left((\sec\beta+\tan\beta)^n-1\right),$$

which is equivalent to showing

$$I_{n+1}+I_{n-1}-2I_n>0.$$

$$\begin{aligned}
&I_{n+1}+I_{n-1}-2I_n\\
&=\int_0^\beta (\sec x+\tan x)^{n+1}\,\mathrm{d}x+\int_0^\beta (\sec x+\tan x)^{n-1}\,\mathrm{d}x-2\int_0^\beta (\sec x+\tan x)^{n}\,\mathrm{d}x\\
&=\int_0^\beta (\sec x+\tan x)^{n-1}\left(2\sec^2 x+2\sec x\tan x-2\sec x-2\tan x\right)\\
&=\int_0^\beta (\sec x+\tan x)^{n-1}\left(\sec^2 x+\tan^2 x+2\sec x\tan x-2\sec x-2\tan x+1\right)\\
&=\int_0^\beta (\sec x+\tan x)^{n-1}\left[(\sec x+\tan x)^2-2\left(\sec x+2\tan x\right)+1\right]\\
&=\int_0^\beta (\sec x+\tan x)^{n-1}\left((\sec x+\tan x)-1\right)^2.
\end{aligned}$$

For $0\le x<\frac{\pi}{2}$, $\sec x>0, \tan x>0$, and so $\sec x+\tan x>0$, $(\sec x+\tan x)^{n-1}>0$.

Also, $\sec x=\frac{1}{\cos x}>\frac{1}{1}=1$, and hence $\sec x+\tan x-1>0$, so $((\sec x+\tan x)-1)^2>0$.

Hence, the integrand is greater than 0 on $(0,\beta)\subseteq\left(0,\frac{\pi}{2}\right)$.

This shows that the desired equation is greater than 0, and hence, we have the desired inequality as desired.

2. Notice that

$$\begin{aligned}
\frac{1}{2}\left(J_{n+1}+J_{n-1}\right) &= \frac{1}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left[(\sec x\cos\beta+\tan x)^2+1\right]\mathrm{d}x \\
&= \frac{1}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left[\sec^2 x\cos^2\beta+\tan^2 x+2\sec x\tan x\cos\beta+1\right]\mathrm{d}x \\
&= \frac{1}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x\cos^2\beta+\sec^2 x+2\sec x\tan x\cos\beta\right)\mathrm{d}x \\
&= \frac{1}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left(2\sec^2 x-\sec^2 x\sin^2\beta+2\sec x\tan x\cos\beta\right)\mathrm{d}x \\
&= \int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x+\sec x\tan x\cos\beta\right)\mathrm{d}x \\
&\quad -\frac{\sin^2\beta}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\sec^2 x\,\mathrm{d}x.
\end{aligned}$$

The first part of the integral integrates similarly:

$$\begin{aligned}
&\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x+\sec x\tan x\cos\beta\right)\mathrm{d}x \\
&= \int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\,\mathrm{d}\left(\sec x\cos\beta+\tan x\right) \\
&= \frac{1}{n}\left[(\sec x\cos\beta+\tan x)^n\right]_0^\beta \\
&= \frac{1}{n}\left[(\sec\beta\cos\beta+\tan\beta)^n-(\sec 0\cos\beta+\tan 0)^n\right] \\
&= \frac{1}{n}\left[(1+\tan\beta)^n-\cos^n\beta\right].
\end{aligned}$$

The second part of the integral has a positive integrand over $(0,\beta)$, and hence the integral is positive, which means

$$\begin{aligned}
\frac{1}{2}\left(J_{n+1}+J_{n-1}\right) &> \int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x+\sec x\tan x\cos\beta\right)\mathrm{d}x \\
&= \frac{1}{n}\left[(1+\tan\beta)^n-\cos^n\beta\right].
\end{aligned}$$

We would like to show that $J_{n+1}+J_{n-1}-2J_n>0$ similar as before to show the final result. Note that

$$\begin{aligned}
&J_{n+1}+J_{n-1}-2J_n \\
&= \int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left[(\sec x\cos\beta+\tan x)^2+1-2\left(\sec x\cos\beta+\tan x\right)\right]\mathrm{d}x \\
&= \int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left[(\sec x\cos\beta+\tan x)-1\right]^2\mathrm{d}x \\
&> 0,
\end{aligned}$$

and hence $J_n<\frac{1}{2}\left(J_{n+1}+J_{n-1}\right)$, which shows

$$J_n<\frac{1}{n}\left((1+\tan\beta)^n-\cos^n\beta\right),$$

as desired.

2021.3 Question 4

1. Since θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, we have

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \mathbf{a}\cdot\mathbf{b}.$$

   Let λ be the angle between $\mathbf{m}$ and $\mathbf{a}$. Hence,

$$\begin{aligned}\cos\lambda &= \frac{\mathbf{a}\cdot\mathbf{m}}{|\mathbf{a}||\mathbf{m}|}\\ &= \frac{\mathbf{a}\cdot\frac{1}{2}(\mathbf{a}+\mathbf{b})}{|\mathbf{m}|}\\ &= \frac{\mathbf{a}\cdot(\mathbf{a}+\mathbf{b})}{|\mathbf{a}+\mathbf{b}|}\\ &= \frac{1+\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}\\ &= \frac{1+\cos\theta}{|\mathbf{a}+\mathbf{b}|}.\end{aligned}$$

   Similarly, let μ be the angle between $\mathbf{m}$ and $\mathbf{b}$, and we must have

$$\cos\lambda = \cos\mu = \frac{1+\cos\theta}{|\mathbf{a}+\mathbf{b}|}.$$

   Since $0 \leq \lambda, \mu \leq \pi$, and cos is one-to-one when restricted to $[0, \pi]$, we must have $\lambda = \mu$, which shows that $\mathbf{m}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

2. We must have $\cos\alpha = \mathbf{a}\cdot\mathbf{c}$, and $\cos\beta = \mathbf{b}\cdot\mathbf{c}$.

   By definition of the projection, we have

$$\begin{aligned}\mathbf{a}_1 &= \mathbf{a} - (\mathbf{a}\cdot\mathbf{c})\,\mathbf{c}\\ &= \mathbf{a} - \cos\alpha\mathbf{c},\end{aligned}$$

   and hence

$$\begin{aligned}\mathbf{a}_1\cdot\mathbf{c} &= \mathbf{a}\cdot\mathbf{c} - \cos\alpha\mathbf{c}\cdot\mathbf{c}\\ &= \cos\alpha - \cos\alpha\\ &= 0,\end{aligned}$$

   as desired.

   Notice that

$$\begin{aligned}|\mathbf{a}_1|^2 &= \mathbf{a}_1\cdot\mathbf{a}_1\\ &= (\mathbf{a} - \cos\alpha\mathbf{c})\cdot(\mathbf{a} - \cos\alpha\mathbf{c})\\ &= \mathbf{a}\cdot\mathbf{a} - 2\cos\alpha\mathbf{a}\cdot\mathbf{c} + \cos^2\alpha\mathbf{c}\cdot\mathbf{c}\\ &= 1 - 2\cos^2\alpha + \cos^2\alpha\\ &= 1 - \cos^2\alpha\\ &= \sin^2\alpha.\end{aligned}$$

   Since $|a_1| \geq 0$, and $0 < \alpha < \frac{\pi}{2}$, $\sin\alpha > 0$, we must have

$$|\mathbf{a}_1| = |\sin\alpha| = \sin\alpha.$$

The angle φ is given by

$$\begin{aligned}\cos\varphi &= \frac{\mathbf{a}_1\cdot\mathbf{b}_1}{|\mathbf{a}_1||\mathbf{b}_1|}\\ &= \frac{(\mathbf{a}-\cos\alpha\mathbf{c})\cdot(\mathbf{b}-\cos\beta\mathbf{c})}{\sin\alpha\sin\beta}\\ &= \frac{\mathbf{a}\cdot\mathbf{b}-\cos\alpha\mathbf{b}\cdot\mathbf{c}-\cos\beta\mathbf{a}\cdot\mathbf{c}+\cos\alpha\cos\beta\mathbf{c}\mathbf{c}}{\sin\alpha\sin\beta}\\ &= \frac{\cos\theta-\cos\alpha\cos\beta-\cos\beta\cos\alpha+\cos\beta\cos\alpha}{\sin\alpha\sin\beta}\\ &= \frac{\cos\theta-\cos\alpha\cos\beta}{\sin\alpha\sin\beta}.\end{aligned}$$

3. By definition of a projection, we have

$$\begin{aligned}\mathbf{m}_1 &= \mathbf{m}-(\mathbf{m}\cdot\mathbf{c})\mathbf{c}\\ &= \frac{1}{2}(\mathbf{a}+\mathbf{b})-\left(\frac{1}{2}(\mathbf{a}+\mathbf{b})\cdot\mathbf{c}\right)\mathbf{c}\\ &= \frac{1}{2}(\mathbf{a}+\mathbf{b})-\left(\frac{1}{2}(\cos\alpha+\cos\beta)\right)\mathbf{c}\\ &= \frac{1}{2}(\mathbf{a}_1+\mathbf{b}_1).\end{aligned}$$

Let ν be the angle between $\mathbf{m}_1$ and $\mathbf{a}_1$, we have

$$\begin{aligned}\cos\nu &= \frac{\mathbf{m}_1\cdot\mathbf{a}_1}{|\mathbf{m}_1||\mathbf{a}_1|}\\ &= \frac{\frac{1}{2}(\mathbf{a}_1+\mathbf{b}_1)\cdot\mathbf{a}_1}{\frac{1}{2}|\mathbf{a}_1+\mathbf{b}_1|\sin\alpha}\\ &= \frac{\mathbf{a}_1\cdot\mathbf{a}_1+\mathbf{b}_1\cdot\mathbf{a}_1}{|\mathbf{a}_1+\mathbf{b}_1|\sin\alpha}\\ &= \frac{\sin^2\alpha+\cos\varphi\sin\alpha\sin\beta}{|\mathbf{a}_1+\mathbf{b}_1|\sin\alpha}\\ &= \frac{\sin^2\alpha+\cos\theta-\cos\alpha\cos\beta}{|\mathbf{a}_1+\mathbf{b}_1|\sin\alpha}.\end{aligned}$$

Similarly, let τ be the angle between $\mathbf{m}_1$ and $\mathbf{b}_1$, we have

$$\cos\tau = \frac{\sin^2\beta+\cos\theta-\cos\alpha\cos\beta}{|\mathbf{a}_1+\mathbf{b}_1|\sin\beta}.$$

Since $0\le\nu,\tau\le\pi$, $\nu=\tau$ if and only if

$$\begin{aligned}\cos\nu &= \cos\tau\\ \frac{\sin^2\alpha+\cos\theta-\cos\alpha\cos\beta}{|\mathbf{a}_1+\mathbf{b}_1|\sin\alpha} &= \frac{\sin^2\beta+\cos\theta-\cos\alpha\cos\beta}{|\mathbf{a}_1+\mathbf{b}_1|\sin\beta}\\ \sin\beta\left(\sin^2\alpha+\cos\theta-\cos\alpha\cos\beta\right) &= \sin\alpha\left(\sin^2\beta+\cos\theta-\cos\alpha\cos\beta\right)\\ \sin\alpha\sin\beta(\sin\alpha-\sin\beta)+\cos\alpha\cos\beta(\sin\alpha-\sin\beta) &= \cos\theta(\sin\alpha-\sin\beta)\\ (\sin\alpha\sin\beta+\cos\alpha\cos\beta)(\sin\alpha-\sin\beta) &= \cos\theta(\sin\alpha-\sin\beta)\\ (\cos(\alpha-\beta)-\cos\theta)(\sin\alpha-\sin\beta) &= 0.\end{aligned}$$

This is if and only if $\sin\alpha=\sin\beta$, or $\cos\theta=\cos(\alpha-\beta)$.

Since $0<\alpha,\beta<\frac{\pi}{2}$, and sin is one-to-one when restricted to $\left(0,\frac{\pi}{2}\right)$, the first condition is true if and only if $\alpha=\beta$.

Hence, $\mathbf{m}_1$ bisects the angle between $\mathbf{a}_1$ and $\mathbf{b}_1$ if and only if $\alpha=\beta$ or $\cos\theta=\cos(\alpha-\beta)$, as desired.

2021.3 Question 5

1. When the curves meet, the r values and the θ values must be both equal, and hence

$$\begin{aligned} a + 2\cos\theta &= 2 + \cos 2\theta \\ a + 2\cos\theta &= 2 + 2\cos^2\theta - 1 \\ 2\cos^2\theta - 2\cos\theta + 1 - a &= 0, \end{aligned}$$

as desired.

By differentiating with respect to theta, for the two curves to touch, we must have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}(a + 2\cos\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}(2 + \cos 2\theta) \\ -2\sin\theta &= -2\sin 2\theta \\ \sin\theta &= \sin 2\theta \\ \sin\theta &= 2\sin\theta\cos\theta \\ \sin\theta(2\cos\theta - 1) &= 0. \end{aligned}$$

This means, either for the value of $\sin\theta = 0$ it satisfies the first equation, or for the value of $2\cos\theta - 1 = 0$ it satisfies the first equation.

For the first case, we must have $\cos\theta = \pm 1$, and hence

$$\begin{aligned} a &= 2\cos^2\theta - 2\cos\theta + 1 \\ &= 2(\pm 1)^2 - 2(\pm 1) + 1 \\ &= 3 \pm 2, \end{aligned}$$

and so $a = 1$ or $a = 5$.

For the second case, we have $\cos\theta = \frac{1}{2}$, and hence

$$\begin{aligned} a &= 2\cos^2\theta - 2\cos\theta + 1 \\ &= 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 1 \\ &= \frac{1}{2}, \end{aligned}$$

as desired.

2. For the case where $a = \frac{1}{2}$, the curves meet precisely for $\cos\theta = \frac{1}{2}$ only, and hence $\theta = \pm\frac{\pi}{3}$, which gives $r = \frac{1}{2} + 1 = \frac{3}{2}$.

Both curves are symmetric about the initial line, since cos is an even function.

When $\theta = 0$, $r_1 = a + 2 = \frac{5}{2}$, and $r_2 = 2 + 1 = 3$.

For r_1, since $r \geq 0$, we must have

$$\begin{aligned} \frac{1}{2} + 2\cos\theta &\geq 0 \\ \cos\theta &\geq -\frac{1}{4}, \end{aligned}$$

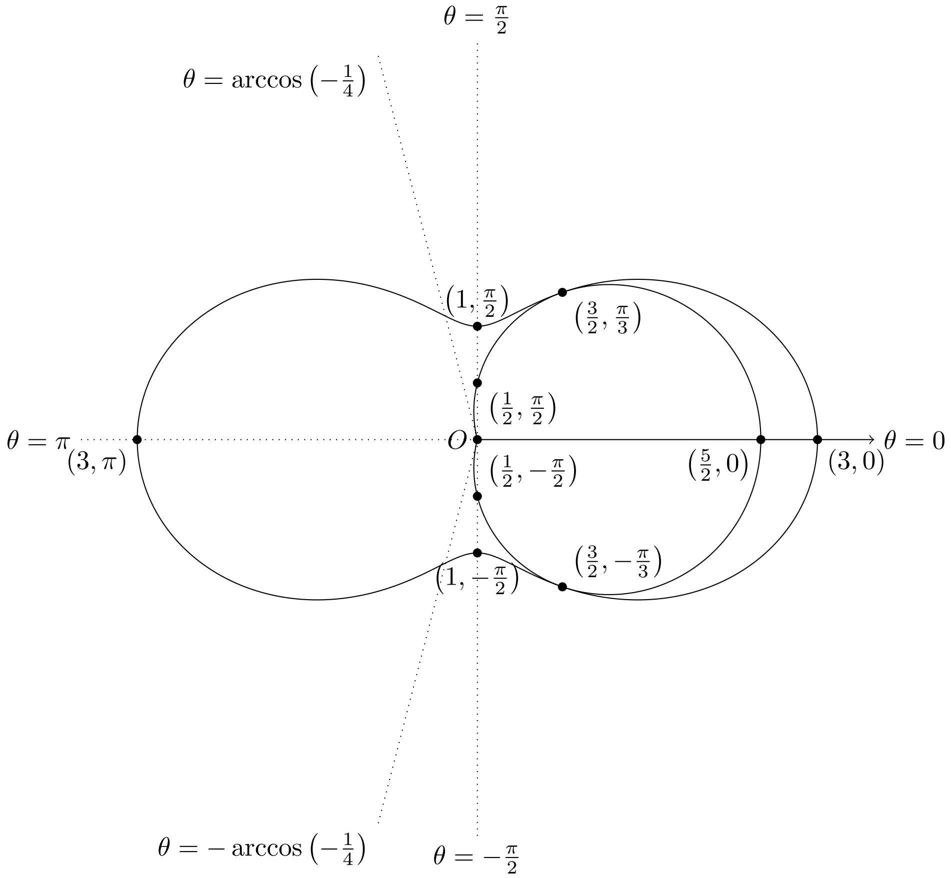
which means it only exists for

$$-\arccos\left(-\frac{1}{4}\right) \leq \theta \leq \arccos\left(-\frac{1}{4}\right).$$

When $\theta = \pm\frac{\pi}{2}$, $r_1 = \frac{1}{2} + 2\cos\pm\frac{\pi}{2} = \frac{1}{2}$.

For all values of θ, we must have $r_2 \geq 0$. When $\theta = \pi$, $r_2 = 2 + 1 = 3$, and for $\theta = \pm\frac{\pi}{2}$, $r_1 = \frac{1}{2} + \cos\pm\frac{\pi}{2} = \frac{1}{2}$, $r_2 = 2 + \cos\pm\pi = 1$.

Hence, the two curves are as follows. All coordinates are in (r, θ).



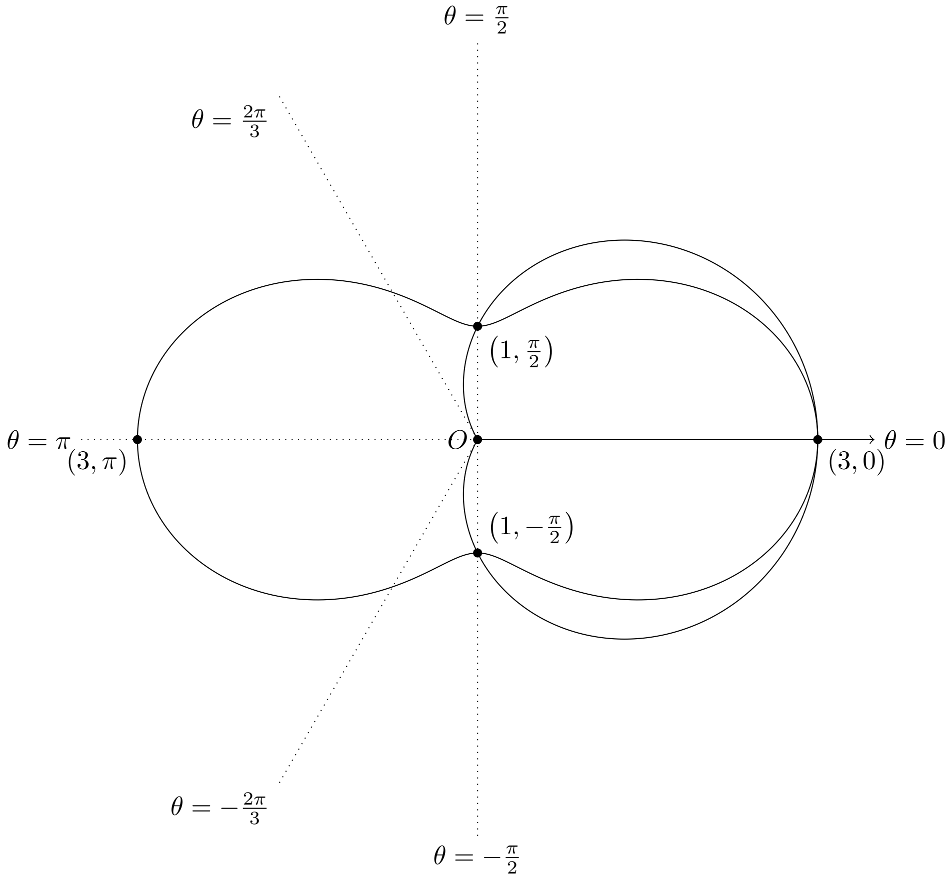
3. • $a = 1$. For r_1, since $r \geq 0$, we must have

$$\begin{aligned} 1 + 2\cos\theta &\geq 0 \\ \cos\theta &\geq -\frac{1}{2}, \end{aligned}$$

which means $-\frac{2}{3}\pi \leq \theta \leq \frac{2}{3}\pi$.
The two curves meet when

$$\begin{aligned} 2\cos^2\theta - 2\cos\theta &= 0 \\ \cos\theta(\cos\theta - 1) &= 0, \end{aligned}$$

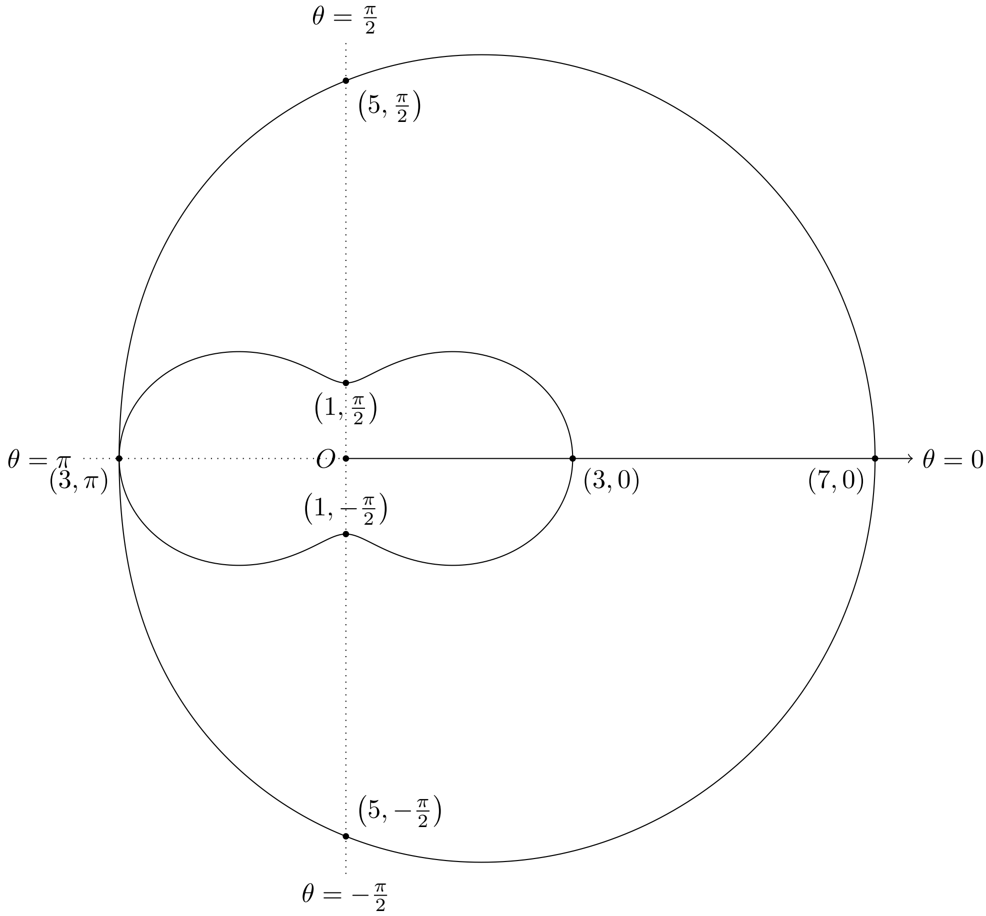
which is when $\cos\theta = 0$ or $\cos\theta = 1$.
For $\cos\theta = 0$, this means $\theta = \pm\frac{\pi}{2}$, and $r = 1$. For this value of θ, the two curves cross.
For $\cos\theta = 1$, this means $\theta = 0$, and $r = 3$. For this value of θ, the two curves touch.



- $a = 5$. For r_1, $r \geq 0$ for all θ.
  The two curves meet when

$$\begin{aligned} 2\cos^2\theta - 2\cos\theta &= 4 \\ \cos^2\theta - \cos\theta - 2 &= 0 \\ (\cos\theta - 2)(\cos\theta + 1) &= 0, \end{aligned}$$

  which is when $\cos\theta = -1$, since $\cos\theta \neq 2$.
  For $\cos\theta = -1$, this means $\theta = \pi$, and $r = 3$. For this value of θ, the two curves touch.
  When $\theta = 0$, $r_1 = 5 + 2 = 7$, and $r_2 = 2 + 1 = 3$. When $\theta = \pm\frac{1}{2}\pi$, $r_1 = 5 + 2\cos\pm\frac{1}{2}\pi = 5$, $r_2 = 2 + \cos\pm\pi = 1$.


$\theta = \frac{\pi}{2}$
$\left(5, \frac{\pi}{2}\right)$
$\left(1, \frac{\pi}{2}\right)$
$\theta = \pi$
$(3, \pi)$
O
$(3, 0)$
$(7, 0)$
$\theta = 0$
$\left(1, -\frac{\pi}{2}\right)$
$\left(5, -\frac{\pi}{2}\right)$
$\theta = -\frac{\pi}{2}$

2021.3 Question 6

1. By multiplying by $\cot\alpha$ on top and bottom of the fraction, we have

$$\begin{aligned} f_\alpha(x) &= \arctan\left(\frac{x+\cot\alpha}{1-x\cot\alpha}\right) \\ &= \arctan\left(\frac{x+\tan\left(\frac{\pi}{2}-\alpha\right)}{1-x\tan\left(\frac{\pi}{2}-\alpha\right)}\right) \\ &= \arctan\tan\left(\arctan x+\frac{\pi}{2}-\alpha\right). \end{aligned}$$

Since $\arctan x \in \left(-\frac{\pi}{2},\alpha\right)\cup\left(\alpha,\frac{\pi}{2}\right)$, we have

$$\arctan x+\frac{\pi}{2}-\alpha \in \left(-\alpha,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi-\alpha\right).$$

Hence, we can simplify this to

$$\begin{aligned} f_\alpha(x) &= \arctan\tan\left(\arctan x+\frac{\pi}{2}-\alpha\right) \\ &= \begin{cases} \arctan x+\frac{\pi}{2}-\alpha, & x<\tan\alpha, \\ \arctan x-\frac{\pi}{2}-\alpha, & x>\tan\alpha. \end{cases} \end{aligned}$$

Hence, by differentiating with respect to x, the constants differentiate to 0, and hence

$$\begin{aligned} f'_\alpha(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\arctan x \\ &= \frac{1}{1+x^2}, \end{aligned}$$

as desired.

The graph consists of 2 branches of arctan, as the simplified expressions suggests. We have the following limiting behaviours of f_α:

$$\begin{aligned} \lim_{x\to-\infty} f_\alpha(x) &= \lim_{x\to-\infty}\arctan x+\frac{\pi}{2}-\alpha=-\alpha, \\ \lim_{x\to\tan\alpha^-} f_\alpha(x) &= \frac{\pi}{2}, \\ \lim_{x\to\tan\alpha^+} f_\alpha(x) &= -\frac{\pi}{2}, \\ \lim_{x\to\infty} f_\alpha(x) &= \lim_{x\to\infty}\arctan x-\frac{\pi}{2}-\alpha=-\alpha, \end{aligned}$$

which shows that f_α has a horizontal asymptote with equation $y=-\alpha$.

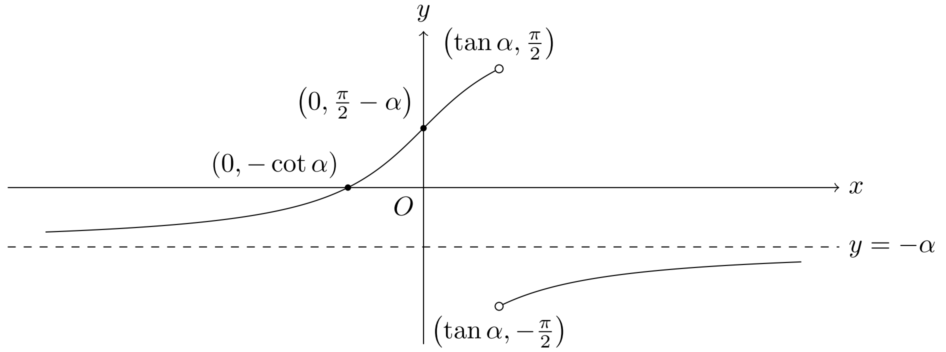
For the intersection with the y-axis,

$$f_\alpha(0)=\arctan 0+\frac{\pi}{2}-\alpha=\frac{\pi}{2}-\alpha,$$

and for the intersection with the x-axis,

$$f_\alpha(x)=0 \iff x\tan\alpha+1=0 \iff x=-\cot\alpha.$$

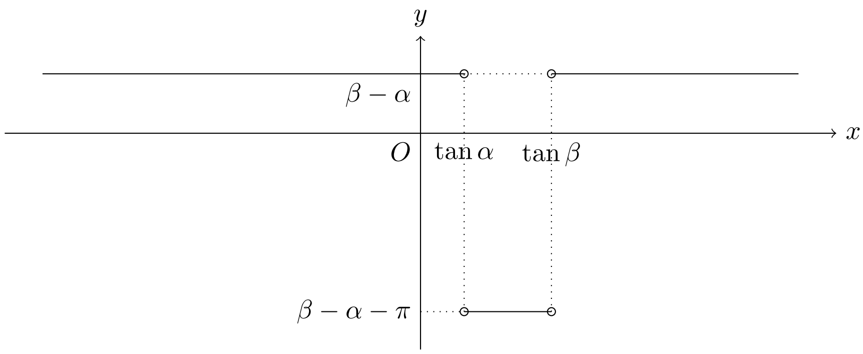
The graph looks as follows.



The domain of this new graph is $x \in \mathbb{R} \setminus \{\tan\alpha, \tan\beta\}$. By considering the functions in the different corresponding ranges, we have

$$f_\alpha(x) - f_\beta(x) = \begin{cases} \left(\arctan(x) + \frac{\pi}{2} - \alpha\right) - \left(\arctan(x) + \frac{\pi}{2} - \beta\right) = \beta - \alpha, & x < \tan\alpha, \\ \left(\arctan(x) - \frac{\pi}{2} - \alpha\right) - \left(\arctan(x) + \frac{\pi}{2} - \beta\right) = \beta - \alpha - \pi, & \tan\alpha < x < \tan\beta, \\ \left(\arctan(x) - \frac{\pi}{2} - \alpha\right) - \left(\arctan(x) - \frac{\pi}{2} - \beta\right) = \beta - \alpha, & \tan\beta < x. \end{cases}$$

Hence, the graph looks as follows.



2. By differentiation, we have

$$\begin{aligned} g'(x) &= \frac{1}{1 - \sin^2 x} \cos x - \frac{1}{\sqrt{1 + \tan^2 x}} \sec^2 x \\ &= \frac{\cos x}{\cos^2 x} - \frac{\sec^2 x}{|\sec x|} \\ &= \sec x - |\sec x| \\ &= \begin{cases} \sec x - \sec x = 0, & 0 \leq x < \frac{1}{2}\pi \text{ or } \frac{3}{2}\pi < x \leq 2\pi, \\ \sec x - (-\sec x) = 2\sec x, & \frac{1}{2}\pi < x < \frac{3}{2}\pi, \end{cases} \end{aligned}$$

since $\sec x$ takes the same sign as $\cos x$, which is negative when $\frac{1}{2}\pi < x < \frac{3}{2}\pi$, and positive when $0 \leq x < \frac{1}{2}\pi$ or $\frac{3}{2}\pi < x \leq 2\pi$ within the range.

For $\frac{1}{2}\pi < x < \frac{3}{2}\pi$, we must have

$$g(x) = \ln|\tan x + \sec x| + C = \ln(-\tan x - \sec x) + C,$$

and by verifying

$$g(\pi) = \operatorname{artanh}(0) - \operatorname{arsinh}(0) = 0,$$

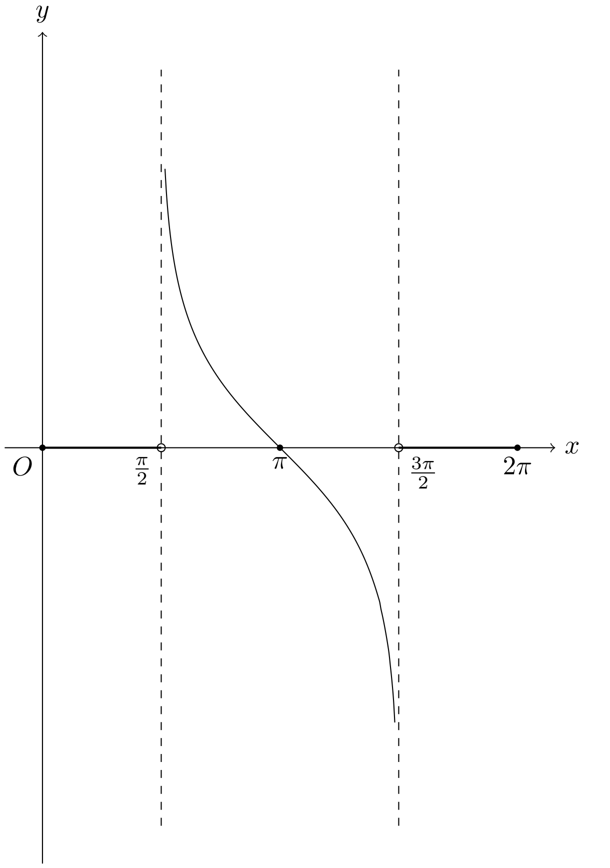
we can see $C = 0$.

Hence, for $0 \leq x < \frac{1}{2}\pi$ and $\frac{3}{2}\pi < x \leq 2\pi$ respectively, $g(x)$ is constant, and notice that

$$g(0) = g(2\pi) = 0,$$

and hence

$$g(x) = \begin{cases} \ln(-\tan x - \sec x), & \frac{1}{2}\pi < x < \frac{3}{2}\pi, \\ 0, & 0 \leq x < \frac{1}{2}\pi \text{ or } \frac{3}{2}\pi \leq 2\pi. \end{cases}$$



y
x
O
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

2021.3 Question 7

1. Notice that

$$\begin{aligned}
z &= \frac{\exp(i\theta)+\exp(i\varphi)}{\exp(i\theta)-\exp(i\varphi)} \\
&= \frac{\exp(i\theta)+\exp(i\varphi)}{\exp(i\theta)-\exp(i\varphi)} \cdot \frac{\exp(-i\theta)-\exp(-i\varphi)}{\exp(-i\theta)-\exp(-i\varphi)} \\
&= \frac{1+\exp(i\varphi-i\theta)-\exp(i\theta-i\varphi)-1}{1-\exp(i\theta-i\varphi)-\exp(i\varphi-i\theta)+1} \\
&= \frac{\exp(i(\varphi-\theta))-\exp(-i(\varphi-\theta))}{2-\exp(-i(\varphi-\theta))-\exp(i(\varphi-\theta))} \\
&= \frac{2i\sin(\varphi-\theta)}{2-2\cos(\varphi-\theta)} \\
&= \frac{i\sin(\varphi-\theta)}{1-\cos(\varphi-\theta)} \\
&= \frac{i\cdot 2\sin\frac{\varphi-\theta}{2}\cos\frac{\varphi-\theta}{2}}{1-(1-2\sin^2\frac{\varphi-\theta}{2})} \\
&= \frac{2i\sin\frac{\varphi-\theta}{2}\cos\frac{\varphi-\theta}{2}}{2\sin^2\frac{\varphi-\theta}{2}} \\
&= i\cot\frac{\varphi-\theta}{2},
\end{aligned}$$

   as desired.

   The modulus of z is $\left|\cot\frac{\varphi-\theta}{2}\right|$. The argument of z is $\pm\frac{\pi}{2}$.

2. Let $a=\exp(i\alpha)$, and $b=\exp(i\beta)$, where $a-b\neq 2n\pi$ for integer n (this ensures that A and B are distinct). We must have $x=a+b=\exp(i\alpha)+\exp(i\beta)$, and $b-a=\exp(i\beta)-\exp(i\alpha)$.

   The vectors representing the two complex numbers are perpendicular, if and only if their argument differ by $\pm\frac{\pi}{2}$, if and only if their ratio has argument $\pm\frac{\pi}{2}$. Notice that the ratios

$$\begin{aligned}
\frac{OX}{AB} &= \frac{a+b}{b-a} \\
&= \frac{\exp(i\alpha)+\exp(i\beta)}{\exp(i\beta)-\exp(i\alpha)}
\end{aligned}$$

   takes the same form as z before, and hence has argument $\pm\frac{\pi}{2}$. This hence means OX is perpendicular to AB.

3. Similarly, let $a=\exp(i\alpha)$, $b=\exp(i\beta)$, and $c=\exp(i\gamma)$, where no pair of α,β and γ differ by some multiple of 2π (which ensures that A,B,C are distinct points).

   If H is the orthocentre of triangle ABC, then

$$h=a+b+c=\exp(i\alpha)+\exp(i\beta)+\exp(i\gamma),$$

   and hence

$$\begin{aligned}
AH &= h-a=b+c=\exp(i\beta)+\exp(i\gamma), \\
BC &= c-b=\exp(i\gamma)-\exp(i\beta).
\end{aligned}$$

   If $h\neq a$, then $AH=b+c\neq 0$, then the angle between AH and BC is given by the argument of the ratio of the complex numbers representing them, and notice

$$\frac{AH}{BC}=\frac{\exp(i\beta)+\exp(i\gamma)}{\exp(i\gamma)-\exp(i\beta)},$$

   which takes the same form of z in the first part. Hence, the argument of this must be $\pm\frac{\pi}{2}$ since $b+c\neq 0$, which shows that AH is perpendicular to BC.

   This means that either $h=a$, or AH is perpendicular to BC, as desired.

4. Similarly, let $a = \exp(i\alpha)$, $b = \exp(i\beta)$, $c = \exp(i\gamma)$ and $d = \exp(i\delta)$, where no pair of α, β, γ and δ differ by some multiple of 2π (which ensures that A, B, C, D are distinct points). Hence,

$$q = b + c + d = \exp(i\beta) + \exp(i\gamma) + \exp(i\delta),$$

and the midpoint of AQ, M, represented by complex number m, is given by

$$m = \frac{a + b + c + d}{2}.$$

By symmetry, the midpoint of BR, CS and DP must also be M.

This means that by an enlargement of scale factor -1 about M, A will be transformed to Q, B to R, C to S, and D to P.

Hence, $ABCD$ is transformed to $PQRS$ by an enlargement of scale factor -1, with centre of enlargement being $\frac{a+b+c+d}{4}$, the midpoint of AQ.

2021.3 Question 8

1. We show this by induction on n.

   We first consider the base case where $n = 1$. Notice LHS $= x_1 = a$, and

   $$\text{RHS} = 2 + 4^{1-1}(a-2) = 2 + (a-2) = a.$$

   Hence, LHS $\geq$ RHS is true.

   Now, assume that the original statement

   $$x_n \geq 2 + 4^{n-1}(a-2)$$

   is true for some $n = k$.

   Consider the case where $n = k + 1$. We first notice that since $a > 2$, we must have

   $$x_n \geq 2 + 4^{n-1}(a-2) > 0.$$

   Hence, we have

   $$\begin{aligned}\text{LHS} &= x_{k+1} \\ &= x_k^2 - 2 \\ &\geq \left(2 + 4^{k-1}(a-2)\right)^2 - 2 \\ &= 4 + 4^{2k-2}(a-2)^2 + 4 \cdot 4^{k-1}(a-2) - 2 \\ &= 2 + 4^k(a-2) + 4^{2k-2}(a-2)^2 \\ &> 2 + 4^{(k+1)-1}(a-2) \\ &= \text{RHS},\end{aligned}$$

   and this shows that the original statement is true for the case $n = k + 1$ as well.

   Hence, the original statement is true for the base case $n = 1$, and given it holds for $n = k$, it holds for $n = k + 1$. By the principle of mathematical induction, it must hold for all integers $n \geq 1$ given $a > 2$, as desired.

2. - **If direction.** We are given that $|a| > 2$. If $a < 0$, we must have $a < -2$, but notice that for $x_1 = a$, $x_2 = a^2 - 2$, and for $x_1 = -a$, $x_2 = (-a)^2 - 2 = a^2 - 2$. Hence, if the first term only differs by a plus/minus sign, all the terms including and after the second term will behave identically. This means we only have to consider the case $a > 2$, and since

     $$x_n \geq 2 + 4^{n-1}(a-2),$$

     and the right-hand side diverges to ∞ as $n \to \infty$, we can conclude that

     $$\lim_{n\to\infty} x_n = \infty,$$

     as desired.

   - **Only-if direction.** We attempt to prove the contrapositive of the only-if direction, i.e. given that $|a| \leq 2$, we want to show that x_n does not diverge to ∞.

     We would like to show that $|x_n| \leq 2$ for all $n \in \mathbb{N}$.

     The base case where $n = 1$ is true, since $0 \leq a \leq 2$.

     Now, assume that this is true for some $n = k$, i.e.

     $$|x_n| \leq 2 \iff -2 \leq x_n \leq 2 \iff 0 \leq x_n^2 \leq 4.$$

     For $n = k + 1$,

     $$x_n = x_{k+1} = x_k^2 - 2,$$

     and hence

     $$-2 \leq x_{k+1} \leq 2 \iff |x_{k+1}| \leq 2.$$

So this statement is true for the base case where $n = 1$, and given it holds for some $n = k$ it holds for the case $n = k+1$. Hence, by the principle of mathematical induction, this statement is true for all $n \in \mathbb{N}$.

This means that x_n is bounded above and below, and hence it cannot diverge to infinity. This proves the contrapositive of the only-if direction, and hence the only-if direction is true.

In conclusion, we have shown that $x_n \to \infty$ as $n \to \infty$ if and only if $|a| > 2$.

3. If this is true for all $n \geq 1$, then this is true for $n = 1$. On one hand,

$$y_1 = \frac{Ax_1}{x_2} = \frac{Aa}{a^2 - 2},$$

and on the other hand

$$y_1 = \frac{\sqrt{x_2^2 - 4}}{x_2} = \frac{\sqrt{(a^2-2)^2 - 4}}{a^2 - 2} = \frac{\sqrt{a^4 - 4a^2}}{a^2 - 2} = \frac{a\sqrt{a^2 - 4}}{a^2 - 2}.$$

Hence, we must have

$$\begin{aligned} A &= \sqrt{a^2 - 4} \\ A^2 &= a^2 - 4 \\ a^2 &= A^2 + 4 \\ a &= \sqrt{A^2 + 4}, \end{aligned}$$

since $a > 2$.

We still have to show that this a gives the desired relation for every $n \geq 1$.

Notice that by definition,

$$\begin{aligned} y_{n+1} &= \frac{A \prod_{i=1}^{n+1} x_i}{x_{n+2}} \\ &= \frac{A \prod_{i=1}^{n}}{x_{n+1}} \cdot \frac{x_{n+1}^2}{x_{n+2}} \\ &= y_n \cdot \frac{x_{n+1}^2}{x_{n+2}}. \end{aligned}$$

We aim to show this by induction on n. The base case where $n = 1$ is shown above.

Now, assume that

$$y_n = \frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$$

for a certain value of $n = k$.

For $n = k + 1$,

$$\begin{aligned} y_n &= y_{k+1} \\ &= y_k \cdot \frac{x_{n+1}^2}{x_{n+2}} \\ &= frac\sqrt{x_{n+1}^2 - 4}x_{n+1} \cdot \frac{x_{n+1}^2}{x_{n+2}} \\ &= \frac{\sqrt{x_{n+1}^2 - 4}x_{n+1}}{x_{n+2}} \\ &= \frac{\sqrt{x_{n+1}^4 - 4x_{n+1}^2}}{x_{n+2}} \\ &= \frac{\sqrt{\left(x_{n+1}^2 - 2\right)^2 - 4}}{x_{n+2}} \\ &= \frac{\sqrt{x_{n+2}^2 - 4}}{x_{n+2}}, \end{aligned}$$

which is precisely the original statement for $n = k + 1$.

By the principle of mathematical induction, for $a = \sqrt{A^2 + 4}$, we have shown that this desired statement holds for the base case $n = 1$, and given that it holds for some $n = k$, we can show it holds for $n = k + 1$. Hence, by the principle of mathematical induction, we have that

$$y_n = \frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$$

for every value of $n \geq 1$ for this certain value of $a = \sqrt{A^2 + 4}$.

Hence, for the value $a = \sqrt{A^2 + 4}$, we have the statement holds for all $n \geq 1$. We have also shown that if the statement holds for all $n \geq 1$, it must be the case that $a = \sqrt{A^2 + 4}$. Hence, for precisely this value of $a = \sqrt{A^2 + 4}$, we have

$$y_n = \frac{\sqrt{x_{n+1}^2 + 4}}{x_{n+1}}.$$

For this value of $a > 2$, we have $x_n \to \infty$ as $n \to \infty$. Hence,

$$y_n = \frac{\sqrt{x_{n+1}^2 + 4}}{x_{n+1}} = \sqrt{1 + \frac{4}{x_{n+1}^2}}$$

converges to 1 as $n \to \infty$.

2021.3 Question 11

1. From the definitions, $X \sim \mathrm{Exp}(\lambda)$, and $Y = \lfloor X \rfloor$.
   Hence, for $n \geq 0$,

$$\begin{aligned}
\mathrm{P}(Y = n) &= \mathrm{P}(\lfloor X \rfloor = n) \\
&= \mathrm{P}(n \leq X < n+1) \\
&= \int_n^{n+1} f(x)\,\mathrm{d}x \\
&= \int_n^{n+1} \lambda \cdot e^{-\lambda x}\,\mathrm{d}x \\
&= \left[-e^{-\lambda x}\right]_n^{n+1} \\
&= -e^{-\lambda(n+1)} + e^{-\lambda n} \\
&= e^{-n\lambda}\left(1 - e^{-\lambda}\right),
\end{aligned}$$

   as desired.

2. Since $Z = X - Y$, we know that $Z = \{X\}$ where $\{x\}$ stands for the fractional part of x.
   Hence, for $0 \leq z \leq 1$, we have

$$\begin{aligned}
\mathrm{P}(Z < z) &= \mathrm{P}(\{X\} < z) \\
&= \mathrm{P}(X - Y < z) \\
&= \sum_{n=0}^{\infty} \mathrm{P}(X < Y + z, Y = n) \\
&= \sum_{n=0}^{\infty} \mathrm{P}(n \leq X < n + z) \\
&= \sum_{n=0}^{\infty} \int_n^{n+z} \lambda \cdot e^{-\lambda x}\,\mathrm{d}x \\
&= \sum_{n=0}^{\infty} \left[-e^{-\lambda x}\right]_n^{n+z} \\
&= \sum_{n=0}^{\infty} \left[-e^{-\lambda(n+z)} + e^{-\lambda n}\right] \\
&= \sum_{n=0}^{\infty} e^{-n\lambda}\left(1 - e^{-\lambda z}\right) \\
&= \left(1 - e^{-\lambda z}\right) \sum_{n=0}^{\infty} e^{-n\lambda} \\
&= \left(1 - e^{-\lambda z}\right) \cdot \frac{1}{1 - e^{-\lambda}} \\
&= \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}},
\end{aligned}$$

   as desired.

3. It must be the case that $0 \leq Z < 1$, and the cumulative distribution function of Z is given by, for $0 \leq z \leq 1$,

$$F_Z(z) = \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}.$$

   By differentiating with respect to z, we get the probability density function of Z is given by, for

$0 \le z \le 1$,

$$\begin{aligned} f_Z(z) &= F_Z'(z) \\ &= \frac{\mathrm{d}}{\mathrm{d}z} \frac{1-e^{-\lambda z}}{1-e^{-\lambda}} \\ &= \frac{1}{1-e^{-\lambda}} \cdot \left(\lambda \cdot e^{-\lambda z}\right) \\ &= \frac{\lambda e^{-\lambda z}}{1-e^{-\lambda}}, \end{aligned}$$

and zero everywhere else.

Hence, the expectation is given by

$$\begin{aligned} \mathrm{E}(Z) &= \int_0^1 z f_Z(z)\,\mathrm{d}z \\ &= \int_0^1 \frac{\lambda z e^{-\lambda z}}{1-e^{-\lambda}}\,\mathrm{d}z \\ &= \frac{\lambda}{1-e^{-\lambda}} \int_0^1 z e^{-\lambda z}\,\mathrm{d}z \\ &= -\frac{1}{1-e^{-\lambda}} \int_0^1 z\,\mathrm{d}e^{-\lambda z} \\ &= -\frac{1}{1-e^{-\lambda}} \left[\left(z e^{-\lambda z}\right)_0^1 - \int_0^1 e^{-\lambda z}\,\mathrm{d}z\right] \\ &= -\frac{1}{1-e^{-\lambda}} \left[z e^{-\lambda z} + \frac{e^{-\lambda z}}{\lambda}\right]_0^1 \\ &= -\frac{1}{1-e^{-\lambda}} \left[\left(e^{-\lambda} + \frac{e^{-\lambda}}{\lambda}\right) - \left(0 + \frac{1}{\lambda}\right)\right] \\ &= \frac{\frac{1}{\lambda} - \frac{e^{-\lambda}}{\lambda} - e^{-\lambda}}{1-e^{-\lambda}} \\ &= \frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda\left(1-e^{-\lambda}\right)}. \end{aligned}$$

4. Since $0 \le z_1 < z_2 \le 1$, we have $n \le n + z_1 < n + z_2 \le n + 1$, and hence

$$\begin{aligned} \mathrm{P}(Y = n, z_1 < Z < z_2) &= \mathrm{P}(Y = n, z_1 < X - Y < z_2) \\ &= \mathrm{P}(n + z_1 < X < n + z_2) \\ &= \int_{n+z_1}^{n+z_2} \lambda \cdot e^{-\lambda x} \\ &= \left[-e^{-\lambda x}\right]_{n+z_1}^{n+z_2} \\ &= e^{-\lambda(n+z_1)} - e^{-\lambda(n+z_2)} \\ &= e^{-\lambda n}\left[e^{-\lambda z_1} - e^{-\lambda z_2}\right]. \end{aligned}$$

On the other hand, notice

$$\begin{aligned} \mathrm{P}(Y = n)\,\mathrm{P}(z_1 < Z < z_2) &= \mathrm{P}(Y = n)\left(\mathrm{P}(Z < z_2) - \mathrm{P}(Z - z_1)\right) \\ &= (1-e^{-\lambda})e^{-n\lambda} \cdot \left[\frac{1-e^{-\lambda z_2}}{1-e^{-\lambda}} - \frac{1-e^{-\lambda z_1}}{1-e^{-\lambda}}\right] \\ &= e^{-n\lambda}\left[\left(1-e^{-\lambda z_2}\right) - \left(1-e^{-\lambda z_1}\right)\right] \\ &= e^{-n\lambda}\left[e^{-\lambda z_1} - e^{-\lambda z_2}\right]. \end{aligned}$$

Hence, we have

$$\mathrm{P}(Y = n, z_1 < Z < z_2) = \mathrm{P}(Y = n)\,\mathrm{P}(z_1 < Z < z_2),$$

and we can conclude that Y and Z are independent.

2021.3 Question 12

1. Let X_i be the outcome of player i in a die roll. Then we have

$$X_{ij} = \begin{cases} 1, & X_i = X_j, \\ 0, & X_i \neq X_j. \end{cases}$$

Hence, we have

$$\begin{aligned} \mathrm{P}(X_{ij} = 1) &= \mathrm{P}(X_i = X_j) \\ &= \sum_{n=1}^{6} \mathrm{P}(X_i = X_j = n) \\ &= \sum_{n=1}^{6} \mathrm{P}(X_i = n)\,\mathrm{P}(X_j = n) \\ &= \sum_{n=1}^{6} \frac{1}{6} \cdot \frac{1}{6} \\ &= 6 \cdot \frac{1}{6} \cdot \frac{1}{6} \\ &= \frac{1}{6}, \end{aligned}$$

and hence $\mathrm{P}(X_{ij} = 0) = 1 - \frac{1}{6} = \frac{5}{6}$. Furthermore,

$$\mathrm{E}\left(X_{ij}\right) = \frac{1}{6} \cdot 1 = \frac{1}{6},$$

and hence

$$\mathrm{Var}\left(X_{ij}\right) = \mathrm{E}\left(X_{ij}^2\right) - \left(X_{ij}\right)^2 = \frac{1}{6} \cdot 1 - \left(\frac{1}{6}\right)^2 = \frac{5}{36}.$$

For any $1 \leq i < j < k \leq n$, we have

$$\begin{aligned} \mathrm{P}(X_{ij} = 1, X_{jk} = 1) &= \mathrm{P}(X_i = X_j, X_j = X_k) \\ &= \mathrm{P}(X_i = X_j = X_k) \\ &= \sum_{n=1}^{6} \mathrm{P}(X_i = X_j = X_k = n) \\ &= \sum_{n=1}^{6} \mathrm{P}(X_i = n)\,\mathrm{P}(X_j = n)\,\mathrm{P}(X_k = n) \\ &= \sum_{n=1}^{6} \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \\ &= 6 \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \\ &= \frac{1}{36} \\ &= \mathrm{P}(X_{ij} = 1)\,\mathrm{P}(X_{jk} = 1), \end{aligned}$$

$$\begin{aligned}
\mathrm{P}(X_{ij}=1,X_{jk}=0)&=\mathrm{P}(X_i=X_j,X_j\neq X_k)\\
&=\sum_{n=1}^{6}\sum_{m\neq n}\mathrm{P}(X_i=X_j=n,X_k=m)\\
&=\sum_{n=1}^{6}\sum_{m\neq n}\mathrm{P}(X_i=n)\,\mathrm{P}(X_j=n)\,\mathrm{P}(X_k=m)\\
&=\sum_{n=1}^{6}\sum_{m\neq n}\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&=6\cdot 5\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&=\frac{5}{36}\\
&=\mathrm{P}(X_{ij}=1)\,\mathrm{P}(X_{jk}=0),
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(X_{ij}=0,X_{jk}=1)&=\mathrm{P}(X_i\neq X_j,X_j=X_k)\\
&=\sum_{n=1}^{6}\sum_{m\neq n}\mathrm{P}(X_i=m,X_j=X_k=m)\\
&=\sum_{n=1}^{6}\sum_{m\neq n}\mathrm{P}(X_i=m)\,\mathrm{P}(X_j=n)\,\mathrm{P}(X_k=n)\\
&=\sum_{n=1}^{6}\sum_{m\neq n}\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&=6\cdot 5\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&=\frac{5}{36}\\
&=\mathrm{P}(X_{ij}=0)\,\mathrm{P}(X_{jk}=1),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{P}(X_{ij}=0,X_{jk}=0)&=\mathrm{P}(X_i\neq X_j,X_j\neq X_k)\\
&=\sum_{n=1}^{6}\sum_{m\neq n}\sum_{l\neq n}\mathrm{P}(X_i=m,X_j=n,X_k=l)\\
&=\sum_{n=1}^{6}\sum_{m\neq n}\sum_{l\neq n}\mathrm{P}(X_i=m)\,\mathrm{P}(X_j=n)\,\mathrm{P}(X_k=l)\\
&=\sum_{n=1}^{6}\sum_{m\neq n}\sum_{l\neq n}\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&=6\cdot 5\cdot 5\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\\
&=\frac{25}{36}\\
&=\mathrm{P}(X_{ij}=0)\,\mathrm{P}(X_{jk}=0).
\end{aligned}$$

Hence, X_{ij} and X_{jk} are independent, and therefore X_{12} is independent of X_{23}.

Similarly, for $0\leq i<j<k\leq n$, we have X_{ij} is independent of X_{ik}, and X_{ik} is independent of X_{jk}. Furthermore, for $0\leq i<j\leq n$ and $0\leq k<p\leq n$, where none of i,j,k,l are equal, we have X_{ij} is independent of X_{kl} since the outcomes are completely irrelevant and independent.

Hence, X_{ij} s are pairwise independent. Let X be the total score:

$$X=\sum_{0\leq i<j\leq n}X_{ij}$$

and hence we have

$$\begin{aligned}\mathrm{E}(X) &= \mathrm{E}\left(\sum_{0\le i<j\le n} X_{ij}\right) \\ &= \sum_{0\le i<j\le n} \mathrm{E}\left(X_{ij}\right) \\ &= \sum_{0\le i<j\le n} \cdot\frac{1}{6} \\ &= \binom{n}{2}\cdot\frac{1}{6} \\ &= \frac{n(n-1)}{12},\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{Var}\left(\sum_{0\le i<j\le n} X_{ij}\right) \\ &= \sum_{0\le i<j\le n} \mathrm{Var}\left(X_{ij}\right) \\ &= \sum_{0\le i<j\le n} \cdot\frac{5}{36} \\ &= \binom{n}{2}\cdot\frac{5}{36} \\ &= \frac{5n(n-1)}{72},\end{aligned}$$

2. Define

$$Y = \sum_{i=1}^{m} Y_i,$$

and hence

$$\mathrm{E}(Y) = \mathrm{E}\left(\sum_{i=1}^{m} Y_i\right) = \sum_{i=1}^{m} \mathrm{E}(Y_i) = 0.$$

Hence,

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{E}\left(Y^2\right) - \mathrm{E}(Y)^2 \\ &= \mathrm{E}\left(\left(\sum_{i=1}^{m} Y_i\right)^2\right) \\ &= \mathrm{E}\left(\sum_{i=1}^{m} Y_i^2 + \sum_{i\ne j} Y_iY_j\right) \\ &= \mathrm{E}\left(\sum_{i=1}^{m} Y_i^2 + 2\sum_{1\le i<j\le m} Y_iY_j\right) \\ &= \mathrm{E}\left(\sum_{i=1}^{m} Y_i^2 + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} Y_iY_j\right) \\ &= \sum_{i=1}^{m} \mathrm{E}\left(Y_i^2\right) + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \mathrm{E}\left(Y_iY_j\right),\end{aligned}$$

as desired.

3. By definition, we have

$$Z_{ij} = \begin{cases} 1, & X_i = X_j \text{ is even,} \\ -1, & X_i = X_j \text{ is odd,} \\ 0, & X_i \neq X_j. \end{cases}$$

Hence, we have $\mathrm{P}(Z_{ij} = 0) = \mathrm{P}(X_{ij} = 0) = \frac{5}{6}$, and

$$\begin{aligned} \mathrm{P}(Z_{ij} = 1) = \mathrm{P}(Z_{ij} = -1) &= \frac{1}{2}\left(1 - \mathrm{P}(Z_{ij} = 0)\right) \\ &= \frac{1}{2}\left(1 - \mathrm{P}(X_{ij} = 0)\right) \\ &= \frac{1}{2}\left(1 - \frac{5}{6}\right) \\ &= \frac{1}{12}, \end{aligned}$$

which means $\mathrm{E}\left(Z_{ij}\right) = 0$.

Consider $Z_{12} = 1$ and $Z_{23} = -1$. If $Z_{12} = 1$ and $Z_{23} = -1$, this means $X_1 = X_2$ are both even, and $X_2 = X_3$ are both odd. This is impossible, and hence

$$\mathrm{P}(Z_{12} = 1, Z_{23} = -1) = 0.$$

On the other hand,

$$\mathrm{P}(Z_{12} = 1)\,\mathrm{P}(Z_{23} = -1) = \frac{1}{12} \cdot \frac{1}{12} = \frac{1}{144} \neq 0,$$

and so Z_{12} and Z_{23} are not independent.

Notice that $X_{ij} = Z_{ij}^2$ and so $\mathrm{E}\left(Z_{ij}^2\right) = \mathrm{E}(X_{ij}) = \frac{1}{6}$.

We can say for $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, where none of i, j, k, l are equal, since X_i, X_j, X_k and X_l are independent, we must have Z_{ij} is independent of Z_{kl}, and hence

$$\mathrm{E}\left(Z_{ij} Z_{kl}\right) = \mathrm{E}\left(Z_{ij}\right) \mathrm{E}\left(Z_{kl}\right) = 0.$$

However, for $1 \leq i < j < k \leq n$, we have

$$\mathrm{P}(Z_{ij} Z_{jk} = -1) = \mathrm{P}(Z_{ij} = 1, Z_{jk} = -1) + \mathrm{P}(Z_{ij} = -1, Z_{jk} = 1) = 0.$$

For the event $Z_{ij} Z_{jk} = 1$, it must be $Z_{ij} = Z_{jk} = \pm 1$, which is the event $X_{ij} = X_{jk} = 1$, and hence

$$\mathrm{P}(Z_{ij} Z_{jk} = 1) = \mathrm{P}(X_{ij} = X_{jk} = 1) = \mathrm{P}(X_{ij} = 1)\,\mathrm{P}(X_{jk} = 1) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

Hence, the only remaining case is $Z_{ij} Z_{jk} = 0$ which gives

$$\mathrm{P}(Z_{ij} Z_{jk} = 0) = 1 - \frac{1}{36} = \frac{35}{36},$$

and hence

$$\mathrm{E}\left(Z_{ij} Z_{jk}\right) = \frac{1}{36}.$$

Let Z be the total score

$$Z = \sum_{1 \leq i < j \leq n} Z_{ij},$$

and hence

$$\mathrm{E}(Z) = \mathrm{E}\left(\sum_{1 \leq i < j \leq n} Z_{ij}\right) = \sum_{1 \leq i < j \leq n} \mathrm{E}\left(Z_{ij}\right) = 0.$$

For the variance, the second part of the sum consists of the non-repeating pairwise products of Z_{ij} and Z_{kl} for $1 \leq i, j, k, l \leq n$, $i < j$ and $k < l$, and finally for non-repeating, $i < k$ or $i = k$ and $j < l$. Let the indices be $1 \leq i < j < k \leq n$, and the pairs must be one of the following three

$$(Z_{ij}, Z_{ik}), (Z_{ij}, Z_{jk}), (Z_{ik}, Z_{jk})$$

and hence there are

$$3 \cdot \binom{n}{3} = \frac{n(n-1)(n-2)}{2}$$

such pairs.

Hence,

$$\begin{aligned}
\mathrm{Var}(Z) &= \sum_{1 \leq i < j \leq n} \mathrm{E}\left(Z_{ij}^2\right) + 2 \cdot \frac{n(n-1)(n-2)}{2} \cdot \frac{1}{36} \\
&= \binom{n}{2} \cdot \frac{1}{6} + \frac{n(n-1)(n-2)}{36} \\
&= \frac{n(n-1)}{12} + \frac{n(n-1)(n-2)}{36} \\
&= \frac{n(n-1)}{36} \cdot [3 + (n-2)] \\
&= \frac{n(n-1)}{36}(n+1) \\
&= \frac{n(n^2-1)}{36},
\end{aligned}$$

as desired.