2020 Paper 3

2020.3 Question 1

1. Using integration by parts, we have

$$\begin{aligned}
I(a,b) &= \int_0^{\frac{\pi}{2}} \cos^a x \cos bx \,\mathrm{d}x \\
&= \frac{1}{b}\int_0^{\frac{\pi}{2}} \cos^a x \,\mathrm{d}\sin bx \\
&= \frac{1}{b}\left[(\cos^a x \sin bx)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin bx \,\mathrm{d}\cos^a x\right] \\
&= -\frac{1}{b}\int_0^{\frac{\pi}{2}} \sin bx \,\mathrm{d}\cos^a x \\
&= \frac{a}{b}\int_0^{\frac{\pi}{2}} \sin bx \sin x \cos^{a-1} x \,\mathrm{d}x.
\end{aligned}$$

Notice that

$$\cos(b-1)x = \cos bx \cos x + \sin bx \sin x,$$

and hence

$$\begin{aligned}
I(a-1,b-1) &= \int_0^{\frac{\pi}{2}} \cos^{a-1} x \cos(b-1)x \,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \cos^{a-1} x(\cos bx \cos x + \sin bx \sin x) \,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \cos^a x \cos bx \,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \sin bx \sin x \cos^{a-1} x \,\mathrm{d}x \\
&= I(a,b) + \frac{b}{a}I(a,b) \\
&= \frac{a+b}{a}I(a,b),
\end{aligned}$$

and hence

$$I(a,b) = \frac{a}{a+b}I(a-1,b-1),$$

as desired.

2. We look at the base case where $n = 0$, and we have

$$\begin{aligned}
\text{LHS} &= \int_0^{\frac{\pi}{2}} \cos(2m+1)x \,\mathrm{d}x \\
&= \frac{1}{2m+1}\left[\sin(2m+1)x\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2m+1}\sin\frac{(2m+1)\pi}{2} \\
&= \frac{(-1)^m}{2m+1},
\end{aligned}$$

and

$$\text{RHS} = (-1)^m \frac{2^0 0!(2m)!m!}{m!(2m+1)!} = \frac{(-1)^m}{2m+1},$$

and so LHS = RHS, which means this holds for the base case where $n = 0$.

Now assume this is true for some $n = k \geq 0$, i.e.

$$I(k, 2m+k+1) = (-1)^m \frac{2^k k!(2m)!(k+m)!}{m!(2k+2m+1)!},$$

and we look at the case $n = k + 1$. Note that

$$\begin{aligned}
\text{LHS} &= I(k+1, 2m+k+2) \\
&= \frac{k+1}{2m+2k+3} I(k, 2m+k+1) \\
&= \frac{k+1}{2m+2k+3} (-1)^m \frac{2^k k!(2m)!(k+m)!}{m!(2k+2m+1)!} \\
&= (-1)^m \frac{2^k k!(2m)!(k+m)!(k+1)}{m!(2k+2m+1)!(2k+2m+3)} \\
&= (-1)^m \frac{2^k (k+1)!(2m)!(k+m)!2(k+m+1)}{m!(2k+2m+1)!(2k+2m+2)(2k+2m+3)} \\
&= (-1)^m \frac{2^{k+1}(k+1)!(2m)!(k+m+1)!}{m!(2k+2m+3)!} \\
&= (-1)^m \frac{2^{k+1}(k+1)!(2m)![(k+1)+m]!}{m!(2(k+1)+2m+1)!},
\end{aligned}$$

which shows the original statement is true for $n = k + 1$.

Hence, by the principle of mathematical induction, the original statement is true for any non-negative integers n, m.

2020.3 Question 2

1. We differentiate with respect to x on both sides, and we have

$$\cosh x + \cosh y \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

If the curve has a stationary point (x, y), we must have $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at that point, and hence

$$\cosh x = 0.$$

This is impossible since the cosh function has a range of $[1, +\infty)$, and hence C has no stationary points. (In fact we must have $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.)
Differentiating this with respect to x again gives

$$\sinh x + \sinh y \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \cosh y \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0.$$

At point (x, y), $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$ if and only if

$$\sinh x + \sinh y \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0.$$

From the previous differentiation, we know that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\cosh x}{\cosh y},$$

and hence

$$\sinh x + \sinh y \cdot \frac{\cosh^2 x}{\cosh^2 y} = 0,$$

which gives

$$\cosh^2 y \sinh x + \sinh y \cosh^2 x = 0.$$

Using the identity $\cosh^2 t = 1 + \sinh^2 t$, we have

$$\sinh x + \sinh^2 y \sinh x + \sinh y + \sinh^2 x \sinh y = 0,$$

and hence

$$(\sinh x + \sinh y)(1 + \sinh x \sinh y) = 0.$$

Since $\sinh x + \sinh y = 2k$ and k is positive, we can conclude that

$$1 + \sinh x \sinh y = 0,$$

as desired.
The only-if direction is identical since all steps above are reversible.
For a point of inflection, we must first have $\frac{\mathrm{d}^y x}{\mathrm{d} =^y} 0$, and hence

$$\sinh x \sinh y = -1, \sinh x + \sinh y = 2k.$$

This means that $\sinh x$ and $\sinh y$ are roots to the quadratic equation in t

$$t^2 - 2kt - 1 = 0.$$

This equation solves to

$$t_{1,2} = \frac{2k \pm \sqrt{4k^2 + 4}}{2} = k \pm \sqrt{k^2 + 1}.$$

Therefore, the points where the second derivative is zero on the curve are

$$\left(\operatorname{arsinh}\left(k \pm \sqrt{k^2 + 1}\right), \operatorname{arsinh}\left(k \mp \sqrt{k^2 + 1}\right)\right).$$

2. If $x + y = a$ and $\sinh x + \sinh y = 2k$, we must have $y = a - x$, and hence

$$\begin{aligned} \frac{e^x - e^{-x}}{2} + \frac{e^{a-x} - e^{x-a}}{2} &= 2k \\ e^{2x} - 1 + e^a - e^{2x-a} &= 4ke^x \\ e^{2x}(1 - e^{-a}) - 4ke^x + (e^a - 1) &= 0, \end{aligned}$$

as desired.

Since e^x is always real, we must have

$$\begin{aligned} (-4k)^2 - 4(1 - e^{-a})(e^a - 1) &= 16k^2 - 4(e^a - 1 - 1 + e^{-a}) \\ &= 16k^2 - 4(2\cosh a - 2) \\ &= 16k^2 + 8 - 8\cosh a \\ &\geq 0, \end{aligned}$$

and hence

$$\cosh a \leq 2k^2 + 1.$$

As for the left-hand side inequality, we already know $\cosh a \geq 1$. $\cosh a = 1$ if and only if $a = x + y = 0$, in which case

$$\sinh x + \sinh y = \sinh x + \sinh(-x) = 0 \neq 2k,$$

since $2k > 0$.

Hence, we must have

$$1 < \cosh a \leq 2k^2 + 1,$$

as desired.

3. Notice that when $\cosh a = 2k^2 + 1$, there is precisely one root to the quadratic equation, which means $x = y$. Hence,

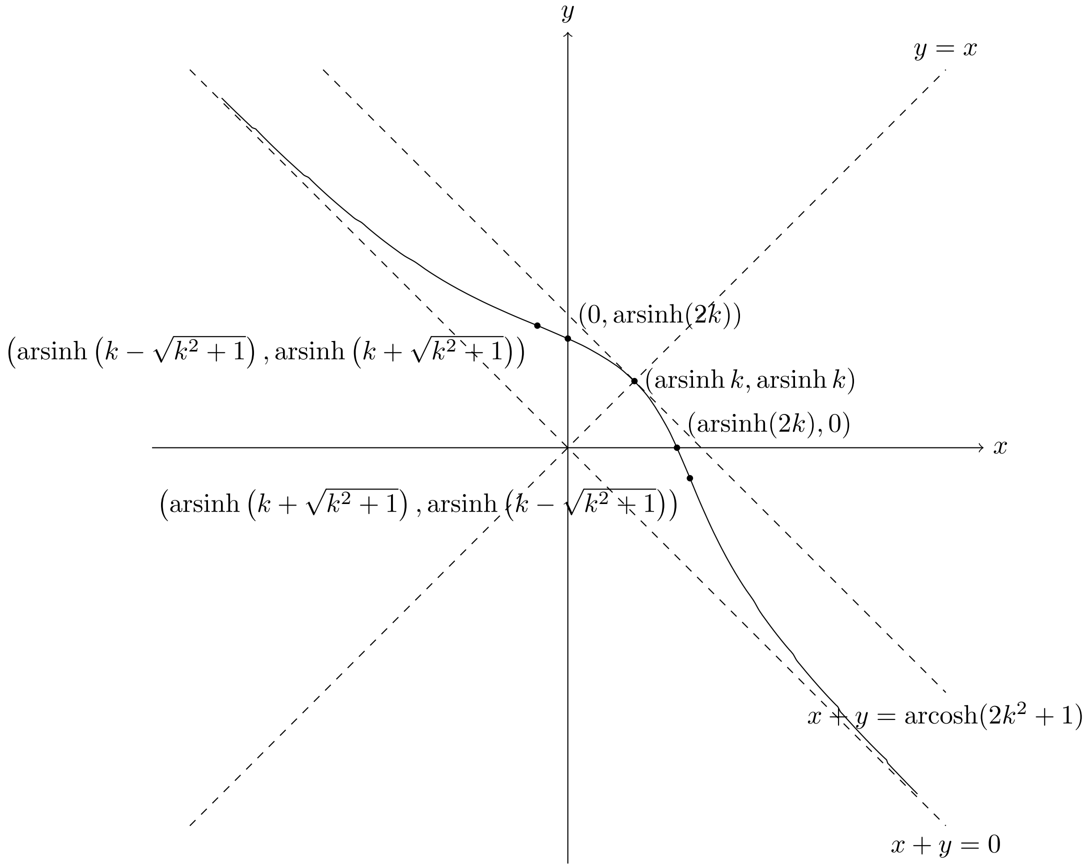
$$\begin{aligned} 2k^2 + 1 &= \cosh a \\ &= \cosh(x + y) \\ &= \cosh x \cosh y + \sinh x \sinh y \\ &= \cosh^2 x + \sinh^2 x \\ &= 1 + 2\sinh^2 x, \end{aligned}$$

which shows that (since $\sinh x + \sinh y = k$)

$$\sinh x = \sinh y = k.$$

The graph meets the axis at $(0, \operatorname{arsinh}(2k))$ and $(\operatorname{arsinh}(2k), 0)$.

Hence, the graph must look as follows:



y
y = x
(0, arsinh(2k))
(arsinh (k − √(k² + 1)) , arsinh (k + √(k² + 1)))
(arsinh k, arsinh k)
(arsinh(2k), 0)
x
(arsinh (k + √(k² + 1)) , arsinh (k − √(k² + 1)))
x + y = arcosh(2k² + 1)
x + y = 0

2020.3 Question 3

1. Let k represent the point K in the complex plane, we must have

$$k - a = (b - a)\exp\left(-i\frac{\pi}{3}\right),$$

and hence

$$k = a + (b - a)\exp\left(-i\frac{\pi}{3}\right).$$

Notice that

$$\omega = \exp\left(\frac{i\pi}{6}\right) = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2},$$

and hence

$$\omega^* = \frac{\sqrt{3}}{2} - i\frac{1}{2}$$

Hence, g_{ab} is given by

$$\begin{aligned} g_{ab} &= \frac{a+b+k}{3} \\ &= \frac{a+b+a+(b-a)\exp\left(-i\frac{\pi}{3}\right)}{3} \\ &= \frac{2a+b+(b-a)\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right]}{3} \\ &= \frac{2a+b+(b-a)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)}{3} \\ &= \frac{\left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)a + \left(\frac{3}{2} - i\frac{\sqrt{3}}{2}\right)b}{3} \\ &= \frac{1}{\sqrt{3}} \cdot \left[\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)a + \left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)b\right] \\ &= \frac{1}{\sqrt{3}} \cdot (\omega a + \omega^* b), \end{aligned}$$

as desired.

2. Q_2 is a parallelogram if and only if

$$\begin{aligned} g_{bc} - g_{ab} &= g_{cd} - g_{da} \\ \frac{1}{\sqrt{3}}(\omega b + \omega^* c) - \frac{1}{\sqrt{3}}(\omega a + \omega^* b) &= \frac{1}{\sqrt{3}}(\omega c + \omega^* d) - \frac{1}{\sqrt{3}}(\omega d + \omega^* a) \\ \omega(b - a - c + d) &= \omega^*(d - a - c + b) \\ (\omega - \omega^*)\left[(b-a) - (c-d)\right] &= 0 \\ (b-a) - (c-d) &= 0 \\ b - a &= c - d, \end{aligned}$$

which is true if and only if Q_1 is a parallelogram. All the steps above are reversible. In particular, $\omega - \omega^* \neq 0$ so we can divide by $\omega - \omega^*$ on both sides.

3. Notice that

$$\begin{aligned} g_{bc} - g_{ab} &= \frac{1}{\sqrt{3}}(\omega b + \omega^* c) - \frac{1}{\sqrt{3}}(\omega a + \omega^* b) \\ &= \frac{1}{\sqrt{3}}\left[\omega^* c - \omega a + (\omega - \omega^*)b\right] \\ &= \frac{1}{\sqrt{3}}\left[\omega^* c - \omega a + bi\right], \end{aligned}$$

and that

$$\begin{aligned}
g_{ca} - g_{ab} &= \frac{1}{\sqrt{3}} \left(\omega c + \omega^* a\right) - \frac{1}{\sqrt{3}} \left(\omega a + \omega^* b\right) \\
&= \frac{1}{\sqrt{3}} \left[\omega c + (\omega^* - \omega) a - \omega^* b\right] \\
&= \frac{1}{\sqrt{3}} \left[\omega c - ai - \omega^* b\right].
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{\omega^*}{\omega} &= \frac{\exp\left(-\frac{i\pi}{6}\right)}{\exp\left(\frac{i\pi}{6}\right)} = \exp\left(-\frac{i\pi}{3}\right), \\
\frac{-\omega}{-i} &= \frac{\omega}{i} = \frac{\exp\left(\frac{i\pi}{6}\right)}{\exp\left(\frac{i\pi}{2}\right)} = \exp\left(-\frac{i\pi}{3}\right), \\
\frac{i}{-\omega^*} &= \frac{-i}{\omega^*} = \frac{\exp\left(-\frac{i\pi}{2}\right)}{\exp\left(-\frac{i\pi}{6}\right)} = \exp\left(-\frac{i\pi}{3}\right),
\end{aligned}$$

and hence we can wee

$$g_{bc} - g_{ab} = (g_{ca} - g_{ab}) \exp\left(-\frac{i\pi}{3}\right),$$

which means G_{BC} is the image of G_{CA} under rotation through $\frac{\pi}{3}$ clockwise about G_{AB}, and this shows that T_2 is an equilateral triangle.

2020.3 Question 4

We first show that Q lies on Π. Notice that

$$\begin{aligned}\mathbf{q}\cdot\mathbf{n} &= (\mathbf{x}-(\mathbf{x}\cdot\mathbf{n})\mathbf{n})\cdot\mathbf{n}\\ &= \mathbf{x}\cdot\mathbf{n}-(\mathbf{x}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{n})\\ &= \mathbf{x}\cdot\mathbf{n}-(\mathbf{x}\cdot\mathbf{n})|\mathbf{n}|^2\\ &= \mathbf{x}\cdot\mathbf{n}-\mathbf{x}\cdot\mathbf{n}\\ &= 0,\end{aligned}$$

so $Q \in \Pi$.

To show PQ is perpendicular to Π, we show that the vector $\overrightarrow{PQ}$ is parallel to the normal vector of Π, and notice

$$\begin{aligned}\overrightarrow{PQ} &= \mathbf{q}-\mathbf{p}\\ &= \mathbf{q}-\mathbf{x}\\ &= -(\mathbf{x}\cdot\mathbf{n})\mathbf{n}\end{aligned}$$

is a scalar multiple of the normal vector $\mathbf{n}$, and so is parallel to $\mathbf{n}$, and perpendicular to Π.

1. The normal vector to this plane is

$$\mathbf{n} = \begin{pmatrix}a\\b\\c\end{pmatrix}.$$

We first find the projection of $\hat{\mathbf{\imath}}$ on Π, and the point's position vector is given by

$$\hat{\mathbf{\imath}}-(\hat{\mathbf{\imath}}\cdot\mathbf{n})\mathbf{n} = \hat{\mathbf{\imath}}-a\mathbf{n}.$$

Hence, the reflection of $\hat{\mathbf{\imath}}$ in Π will be the point with position vector

$$\begin{aligned}\hat{\mathbf{\imath}}+2\left[(\hat{\mathbf{\imath}}-a\mathbf{n})-\hat{\mathbf{\imath}}\right] &= \hat{\mathbf{\imath}}-2\left[a\mathbf{n}\right]\\ &= \begin{pmatrix}1-2a^2\\-2ab\\-2ac\end{pmatrix}\\ &= \begin{pmatrix}a^2+b^2+c^2-2a^2\\-2ab\\-2ac\end{pmatrix}\\ &= \begin{pmatrix}b^2+c^2-a^2\\-2ab\\-2ac\end{pmatrix},\end{aligned}$$

as desired.

Similarly, the image of $\hat{\mathbf{\jmath}}$ under T is

$$\begin{pmatrix}-2ab\\a^2+c^2-b^2\\-2bc\end{pmatrix},$$

and the image of $\hat{\mathbf{k}}$ under T is

$$\begin{pmatrix}-2ac\\-2bc\\a^2+b^2-c^2\end{pmatrix}.$$

Hence, the matrix $\mathbf{M}$ which represents T is given by

$$\mathbf{M} = \begin{pmatrix}b^2+c^2-a^2 & -2ab & -2ac\\ -2ab & a^2+c^2-b^2 & -2bc\\ -2ac & -2bc & a^2+b^2-c^2\end{pmatrix}.$$

2. Since the elements on the diagonal of $\mathbf{M}$ are given by $1-2a^2, 1-2b^2, 1-2c^2$, we must have

$$1-2a^2=0.64, 1-2b^2=0.36, 1-2c^2=0,$$

and hence

$$a^2=\frac{9}{50}, b^2=\frac{8}{25}, c^2=\frac{1}{2},$$

which gives

$$a=\pm\frac{3}{5\sqrt{2}}, b=\pm\frac{2\sqrt{2}}{5}, c=\pm\frac{1}{\sqrt{2}},$$

here the $\pm$ signs do not necessarily match up.

Since $-2ab>0$, $-2ac>0$, $-2bc<0$, we have a and b, a and c take different signs, and b and c take the same sign.

Hence,

$$(a,b,c)=\left(\pm\frac{3}{5\sqrt{2}}, \mp\frac{2\sqrt{2}}{5}, \mp\frac{1}{\sqrt{2}}\right).$$

We verify these triples indeed satisfy the non-diagonal elements as well.

The Cartesian equation of the plane is therefore given by

$$\begin{aligned} ax+by+cz&=0\\ \frac{3}{5\sqrt{2}}x-\frac{2\sqrt{2}}{5}y-\frac{1}{\sqrt{2}}z&=0\\ 3x-4y-5z&=0. \end{aligned}$$

3. The line has equation

$$l:\mathbf{r}=\lambda\mathbf{n}, \lambda\in\mathbb{R}.$$

A rotation about a line through π is simply a 'perpendicular' reflection of the point about the line, i.e. the reflection in the point on the line such that the point and the original point is perpendicular to the line.

Let the original point be P with position vector $\mathbf{x}$, and let the new point be Q with position vector $\lambda\mathbf{n}$, we must have

$$(\lambda\mathbf{n}-\mathbf{x})\cdot\mathbf{n}=\lambda\mathbf{n}\cdot\mathbf{n}-\mathbf{x}\cdot\mathbf{n}=0,$$

which means

$$\lambda=\mathbf{x}\cdot\mathbf{n},$$

and

$$\mathbf{q}=(\mathbf{x}\cdot\mathbf{n})\mathbf{n}.$$

Hence, the image of P under this transformation is

$$\begin{aligned} \mathbf{p}+2(\mathbf{q}-\mathbf{p})&=2\mathbf{q}-\mathbf{p}\\ &=2(\mathbf{x}\cdot\mathbf{n})\mathbf{n}-\mathbf{x}. \end{aligned}$$

If $\mathbf{x}=\hat{\mathbf{i}}$, the image is

$$2a\mathbf{n}-\hat{\mathbf{i}}=\begin{pmatrix}2a^2-1\\2ab\\2ac\end{pmatrix}=\begin{pmatrix}a^2-b^2-c^2\\2ab\\2ac\end{pmatrix}.$$

Similarly, the image of $\hat{\mathbf{j}}$ under this transformation is

$$\begin{pmatrix}2ab\\b^2-a^2-c^2\\2bc\end{pmatrix},$$

and the image of $\hat{\mathbf{k}}$ under this transformation is

$$\begin{pmatrix} 2ac \\ 2bc \\ c^2 - a^2 - b^2 \end{pmatrix}.$$

Hence, the matrix which represents this transformation, $\mathbf{N}$, is given by

$$\mathbf{N} = \begin{pmatrix} a^2 - b^2 - c^2 & 2ab & 2ac \\ 2ab & b^2 - a^2 - c^2 & 2bc \\ 2ac & 2bc & c^2 - a^2 - b^2. \end{pmatrix}$$

4. Notice that since $\mathbf{N} = -\mathbf{M}$, and $\mathbf{M}$ by definition is self-inverse, we have

$$\mathbf{NM} = -\mathbf{MM} = -\mathbf{M}^2 = -\mathbf{I},$$

which is an enlargement of scale factor (-1) with the centre being the origin.

2020.3 Question 5

We notice that

$$\begin{aligned}
\text{RHS} &= (x-y)\sum_{r=1}^{n} x^{n-r}y^{r-1} \\
&= x\sum_{r=1}^{n} x^{n-r}y^{r-1} - y\sum_{r=1}^{n} x^{n-r}y^{r-1} \\
&= \sum_{r=1}^{n} x^{n-r+1}y^{r-1} - \sum_{r=1}^{n} x^{n-r}y^{r} \\
&= \sum_{r=0}^{n-1} x^{n-r}y^{r} - \sum_{r=1}^{n} x^{n-r}y^{r} \\
&= x^n y^0 + \sum_{r=1}^{n-1} x^{n-r}y^{r} - \sum_{r=1}^{n-1} x^{n-r}y^{r} - x^0 y^n \\
&= x^n - y^n.
\end{aligned}$$

1. Notice that

$$\begin{aligned}
f(x) &= x^n \cdot \left(F(x) - \frac{A}{x-k}\right) \\
&= x^n \cdot \left(\frac{1}{x^n(x-k)} - \frac{A}{x-k}\right) \\
&= \frac{1}{x-k} - \frac{Ax^n}{x-k} \\
&= \frac{1-Ax^n}{x-k}.
\end{aligned}$$

Since f is a polynomial, the numerator must be divisible by the denominator, and hence when $x = k$, the numerator must be 0, which means

$$1 - Ak^n = 0,$$

and hence

$$A = \frac{1}{k^n}.$$

Hence,

$$f(x) = \frac{1-Ax^n}{x-k} = \frac{1}{x-k}\left(1 - \left(\frac{x}{k}\right)^n\right),$$

as desired.

Using the identity, we have

$$\begin{aligned}
f(x) &= \frac{1-Ax^n}{x-k} \\
&= \frac{1^n - \left(\frac{x}{k}\right)^n}{x-k} \\
&= \frac{1}{k^n} \cdot \frac{k^n - x^n}{x-k} \\
&= \frac{1}{k^n} \cdot \frac{-(x-k)\sum_{r=1}^{n} k^{n-r}x^{r-1}}{x-k} \\
&= -\sum_{r=1}^{n} k^{-r}x^{r-1},
\end{aligned}$$

and hence

$$\begin{aligned}F(x) &= \frac{A}{x-k} + \frac{f(x)}{x^n} \\ &= \frac{1}{k^n(x-k)} - \frac{\sum_{r=1}^{n} k^{-r} x^{r-1}}{x^n} \\ &= \frac{1}{k^n(x-k)} - \sum_{r=1}^{n} \frac{1}{k^r x^{n-r+1}} \\ &= \frac{1}{k^n(x-k)} - \sum_{r=1}^{n} \frac{1}{k^{n-r+1} x^r} \\ &= \frac{1}{k^n(x-k)} - \frac{1}{k} \sum_{r=1}^{n} \frac{1}{k^{n-r} x^r},\end{aligned}$$

as desired.

2. Notice that on one hand,

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n F(x) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x-k} = -\frac{1}{(x-k)^2},$$

and on the other hand, using the expression above, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} x^n F(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{x^n}{k^n(x-k)} - \frac{1}{k} \sum_{r=1}^{n} \frac{x^{n-r}}{k^{n-r}} \right] \\ &= \frac{n x^{n-1} k^n (x-k) - x^n k^n}{k^{2n}(x-k)^2} - \frac{1}{k} \sum_{r=1}^{n} \frac{(n-r) x^{n-r-1}}{k^{n-r}} \\ &= \frac{n x^{n-1}(x-k) - x^n}{k^n (x-k)^2} - \sum_{r=1}^{n} \frac{n-r}{k^{n-r+1} x^{-n+r+1}} \\ &= \frac{n x^{n-1}}{k^n(x-k)} - \frac{x^n}{k^n(x-k)^2} - \sum_{r=1}^{n} \frac{n-r}{k^{n-r+1} x^{-n+r+1}}.\end{aligned}$$

Hence,

$$\begin{aligned}-\frac{1}{(x-k)^2} &= \frac{n x^{n-1}}{k^n(x-k)} - \frac{x^n}{k^n(x-k)^2} - \sum_{r=1}^{n} \frac{n-r}{k^{n-r+1} x^{-n+r+1}} \\ \frac{1}{(x-k)^2} &= \frac{x^n}{k^n(x-k)^2} - \frac{n x^{n-1}}{k^n(x-k)} + \sum_{r=1}^{n} \frac{n-r}{k^{n-r+1} x^{-n+r+1}} \\ \frac{1}{x^n(x-k)^2} &= \frac{1}{k^n(x-k)^2} - \frac{n}{k^n x(x-k)} + \sum_{r=1}^{n} \frac{n-r}{k^{n-r+1} x^{r+1}},\end{aligned}$$

precisely as desired.

3. Let $n = 3$ and $k = 1$, and hence we have

$$\begin{aligned}\frac{1}{x^3(x-1)^2} &= \frac{1}{(x-1)^2} - \frac{3}{x(x-1)} + \sum_{r=1}^{3} \frac{3-r}{x^{r+1}} \\ &= \frac{1}{(x-1)^2} - \frac{3}{x-1} + \frac{3}{x} + \frac{2}{x^2} + \frac{1}{x^3}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\int_2^N \frac{\mathrm{d}x}{x^3(x-1)^2} &= \int_2^N \left[\frac{1}{(x-1)^2} - \frac{3}{x-1} + \frac{3}{x} + \frac{2}{x^2} + \frac{1}{x^3}\right] \mathrm{d}x \\ &= \left[-\frac{1}{x-1} - 3\ln|x-1| + 3\ln|x| - \frac{2}{x} - \frac{1}{2x^2}\right]_2^N \\ &= \left(3\ln\frac{N}{N-1} - \frac{1}{N-1} - \frac{2}{N} - \frac{1}{2N^2}\right) - \left(3\ln\frac{2}{1} - \frac{1}{1} - \frac{2}{2} - \frac{1}{2\cdot 4}\right) \\ &= \frac{17}{8} - 3\ln 2 + \left(3\ln\frac{N}{N-1} - \frac{1}{N-1} - \frac{2}{N} - \frac{1}{2N^2}\right).\end{aligned}$$

We take the limit as $N \to \infty$. Since $\frac{N}{N-1} = 1 + \frac{1}{N-1}$, and as $N \to \infty$, $\frac{1}{N-1} \to 0$, this means that $\ln\frac{N}{N-1} \to 0$. All the fractions with N on the denominator also approaches 0. Hence, the limit of this integral as $N \to \infty$ is

$$\int_2^\infty \frac{\mathrm{d}x}{x^3(x-1)^2} = \frac{17}{8} - 3\ln 2.$$

2020.3 Question 6

1. Note that this function has symmetry about the y-axis since cos is an even function.

   When $x = 0$, $y = 1 + \sqrt{1} = 2$. When $x = \pm\frac{\pi}{4}$, $y = \frac{1}{\sqrt{2}}$.

   We investigate the gradient:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin x - 2\sin 2x \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\cos 2x}} \\ &= -\sin x - \frac{\sin 2x}{\sqrt{\cos 2x}},\end{aligned}$$

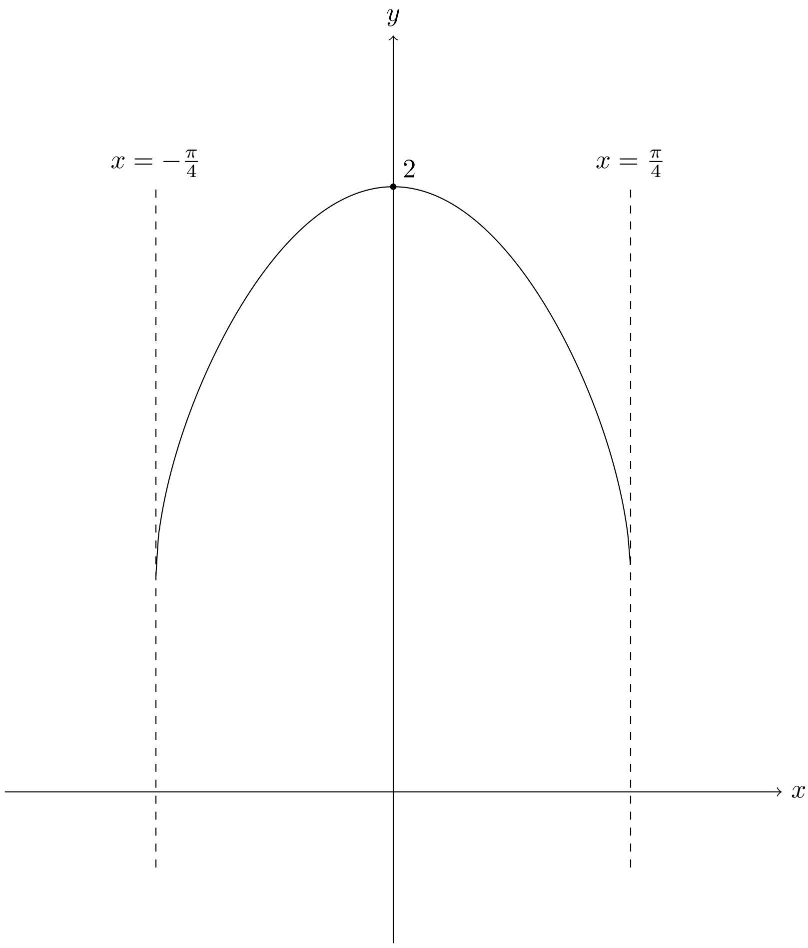
   so $\frac{\mathrm{d}y}{\mathrm{d}x}$ takes opposite sign as x, which means that y is decreasing when $x > 0$, and y is increasing when $x < 0$, and $x = 0$ gives a maximum.

   Also, note that

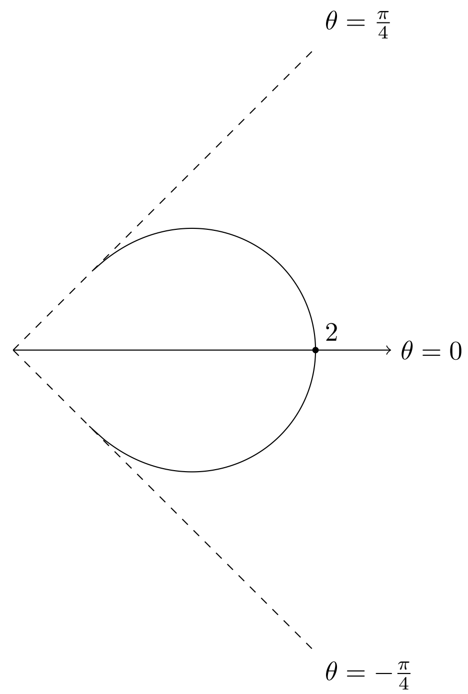
$$\lim_{x \to \frac{\pi}{4}^-} \frac{\mathrm{d}y}{\mathrm{d}x} = -\infty, \lim_{x \to -\frac{\pi}{4}^+} \frac{\mathrm{d}y}{\mathrm{d}x} = \infty,$$

   which means the tangent to the graph at those points are vertical.

   Hence, the graph looks as follows:



2. The graph looks as follows.



3. By solving the quadratic, we have

$$r = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4\sin^2\theta}}{2} = \cos\theta \pm \sqrt{\cos 2\theta}.$$

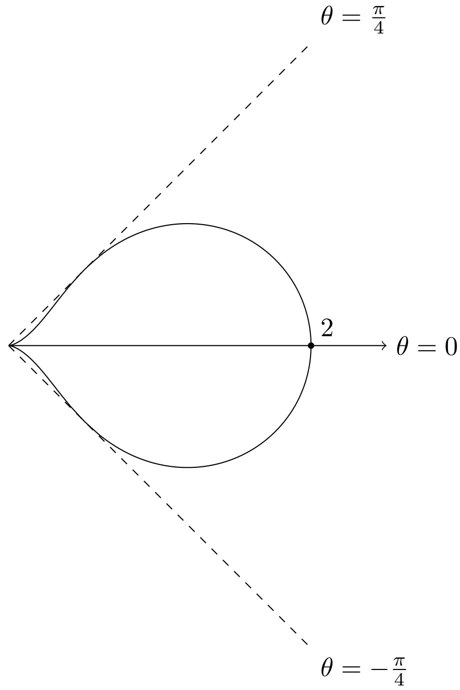
Hence, at $\theta = \pm\frac{1}{4}\pi, r = \frac{1}{\sqrt{2}}$.

When r is small, we must have that $r = \cos\theta - \sqrt{\cos 2\theta}$ and θ is small, and

$$\begin{aligned}-2r\cos\theta + \sin^2\theta &\approx 0\\ r &\approx \frac{\sin^2\theta}{2\cos\theta}\\ r &\approx \frac{1}{2}\sin\theta\tan\theta\\ r &\approx \frac{1}{2}\theta^2,\end{aligned}$$

as desired.

The curve will look as follows. At $\theta = \pm\frac{1}{4}\pi$, the curve is tangent to the lines. At $r = 0$, the curves are tangent to the initial line.



The area between C_2 and $\theta = 0$ above the line is given by

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[\left(\cos\theta + \sqrt{\cos 2\theta}\right)^2 - \left(\cos\theta - \sqrt{\cos 2\theta}\right)^2\right] \mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{4}} 4\cos\theta\sqrt{\cos 2\theta}\,\mathrm{d}\theta \\
&= 2\int_0^{\frac{\pi}{4}} \cos\theta\sqrt{\cos 2\theta}\,\mathrm{d}\theta \\
&= 2\int_0^{\frac{\pi}{4}} \cos\theta\sqrt{1-2\sin^2\theta}\,\mathrm{d}\theta \\
&= 2\int_0^{\frac{\pi}{4}} \sqrt{1-2\sin^2\theta}\,\mathrm{d}\sin\theta \\
&= 2\int_0^{\frac{1}{\sqrt{2}}} \sqrt{1-2x^2}\,\mathrm{d}x \\
&= \sqrt{2}\int_0^1 \sqrt{1-y^2}\,\mathrm{d}y \\
&= \sqrt{2}\cdot\frac{\pi}{4} \\
&= \frac{\pi}{2\sqrt{2}},
\end{aligned}$$

as desired, the final integral being because this is $\frac{1}{4}$ of the area of the unit circle, which is π.

2020.3 Question 7

1. By differentiating both sides of the second differential equation, we can see

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+g'(x)y+g(x)\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{\mathrm{d}u}{\mathrm{d}x}\\&=h(x)-f(x)u\\&=h(x)-f(x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}+g(x)y\right),\end{aligned}$$

   and hence rearranging gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(f(x)+g(x))\frac{\mathrm{d}y}{\mathrm{d}x}+(g'(x)+f(x)g(x))y=f(x),$$

   as desired.

2. We must have $g(x)+f(x)=1+\frac{4}{x}$, and $g'(x)+f(x)g(x)=\frac{2}{x}+\frac{2}{x^2}$, with $h(x)=4x+12$.

   Hence, from the first equation, we have $f(x)=1+\frac{4}{x}-g(x)$, and putting this into the second equation gives us

$$g'(x)+\left(1+\frac{4}{x}-g(x)\right)g(x)=\frac{2}{x}+\frac{2}{x^2}.$$

   If $g(x)=kx^n$, then $g'(x)=knx^{n-1}$, and putting this back we have

$$knx^{n-1}+\left(1+\frac{4}{x}-kx^n\right)kx^n=\frac{2}{x}+\frac{2}{x^2},$$

   which gives

$$-k^2x^{2n}+kx^n+k(n+4)x^{n-1}=2x^{-1}+2x^{-2}.$$

   Therefore, we could simply let $n=-1$, and $k=2$. Verify that

$$\text{LHS}=-4x^{-2}+2x^{-1}+2\cdot 3x^{-2}=2x^{-1}+2x^{-2}=\text{RHS}.$$

   Hence, $g(x)=\frac{2}{x}$, and $f(x)=1+\frac{2}{x}$.

   The differential equation for u is

$$\frac{\mathrm{d}u}{\mathrm{d}x}+\left(1+\frac{2}{x}\right)u=4x+12.$$

   The integration factor is

$$I(x)=e^{\int(1+\frac{2}{x})\,\mathrm{d}x}=e^{x+2\ln x}=x^2e^x,$$

   and hence

$$x^2e^x\frac{\mathrm{d}u}{\mathrm{d}x}+e^x(x^2+2x)u=\frac{\mathrm{d}x^2e^xu}{\mathrm{d}x}=4x^3e^x+12x^2e^x.$$

   Notice the right-hand side is the derivative of $4x^3e^x$, and hence

$$x^2e^xu=4x^3e^x+C.$$

   When $x=1$,

$$\begin{aligned}u|_{x=1}&=\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1}+g(1)\,y|_{x=1}\\&=-3+2\cdot 5\\&=7,\end{aligned}$$

   and hence

$$7e=4e+C,$$

giving $C = 3e$.

Hence,

$$u = 4x + \frac{3e}{x^2 e^x}.$$

The differential equation for y gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = u,$$

and hence the integration factor is x^2, giving

$$\frac{\mathrm{d}x^2 y}{\mathrm{d}x} = 4x^3 + 3e \cdot e^{-x},$$

and hence by integration on both sides, we have

$$x^2 y = x^4 - 3e \cdot e^{-x} + C'.$$

Since when $x = 1$, $y = 5$, we must have

$$5 = 1 - 3 + C',$$

giving $C' = 7$.

Hence,

$$x^2 y = x^4 - 3e^{1-x} + 7,$$

and hence

$$y = x^2 - 3x^{-2}e^{1-x} + 7x^{-2}.$$

2020.3 Question 8

1. It is not difficult to see that the terms in the sequence are all positive. Consecutive pairs of terms with the first one having odd index in the sequence are u_{2k+1} and u_{2k+2} for $k \geq 1$ (the case where $k = 0$ is excluded due to the first pair being separately considered). We have

$$\begin{aligned} u_{2k+1} - u_{2k+2} &= (u_k + u_{k+1}) - u_{k+1} \\ &= u_k \\ &> 0, \end{aligned}$$

so $u_{2k+1} > u_{2k+2}$.

Consider the terms u_{2k} and u_{2k+1} for $k \geq 1$, which is consecutive pairs of terms with the first one having even index. We notice

$$\begin{aligned} u_{2k+1} - u_{2k} &= (u_k + u_{k+1}) - u_k \\ &= u_{k+1} \\ &> 0, \end{aligned}$$

so $u_{2k+1} > u_{2k}$.

As for the first pair $u_1 = u_2 = 1$, the term with the odd index is not greater than the term of even index.

Hence, for every pair of consecutive terms of this sequence, except the first pair, the term with odd subscript is larger than the term with even subscript, as desired.

2. If the two consecutive terms take the form $u_{2k+1} = u_k + u_{k+1}$ and $u_{2k+2} = u_{k+1}$, we have $u_k = u_{2k+1} - u_{2k+2}$. If $d \mid u_{2k+1}$ and $d \mid u_{2k+2}$, we must have $d \mid u_k = u_{2k+1} - u_{2k+2}$, and $d \mid u_{k+1} = u_{2k+2}$, which are two consecutive terms as well. Notice that $k + 1 < 2k + 2$ for $k \geq 1$, so this is some pair before the original pair.

In the other case where the two consecutive terms take the form $u_{2k} = u_k$ and $u_{2k+1} = u_k + u_{k+1}$, we have $u_{k+1} = u_{2k+1} - u_{2k}$. If $d \mid u_{2k}$ and $d \mid u_{2k+1}$, we must have $d \mid u_{k+1} = u_{2k+1} - u_{2k}$, and $d \mid u_k = u_{2k}$, which are two consecutive terms as well. Notice that $k+1 < 2k+1$ for $k \geq 1$, so this is some pair before the original pair.

We use the idea of proof by infinite descent in this part. The first two terms $u_1 = u_2 = 1$ are co-prime, since one is the only common factor they share. Now, assume there exists some pair of consecutive terms in the sequence that are not co-prime, then there is one with the smallest pair of indices.

If this pair is the first two terms, this is impossible since the first two terms are co-prime. If they are not, by the previous part, there must exist another pair of consecutive terms with smaller indices, which contradicts with this pair being the pair with the smallest indices.

Hence, such pair of consecutive terms in the sequence being not co-prime does not exist.

3. We still use the idea of proof by infinite descent here. B.W.O.C assume that two integers appear consecutively in the same order twice. We consider the first pair of consecutive integers appearing twice, with the smallest indices. There are two cases:

   - The indices where they appear are u_{2k} and u_{2k+1}, where $k \geq 1$. Let

     $$u_{2k} = c, u_{2k+1} = d,$$

     and hence $d > c$.

     Since they must re-appear in the same order, it must be the case that they re-appear at $u_{2m} = c$ and $u_{2m+1} = d$, since the odd-indexed term is always greater than the even-indexed term, and here $m > n$.

     Since

     $$u_{2k} = u_k = c, u_{2k+1} = u_k + u_{k+1} = d,$$

     we have

     $$u_{k+1} = d - c,$$

and similarly
$$u_{m+1} = d - c.$$

So $(u_k, u_{k+1}) = (u_m, u_{m+1}) = (c, d - c)$. But since $k \geq 1$, $2k > k$, and this implies that u_{2k} and u_{2k+1} is not the first pair of consecutive integers appearing twice, hence leading to a contradiction.

- The indices where they appear are u_{2k-1} and u_{2k}, where $k \geq 1$. Let

$$u_{2k-1} = c, u_{2k} = d,$$

and hence $c \geq d$ with the equal sign taking place if and only if $k = 1$. By similar reasoning, it must be the case for some $m > k$ that $u_{2m-1} = c$ and $u_{2m} = d$. Since $m > k \geq 1$, we must have $c > d$ and hence $k > 1$. Hence,

$$u_{2k-1} = u_{k-1} + u_k = c, u_{2k} = u_k = d,$$

implying
$$u_{k-1} = u_{2k-1} - u_{2k} = c - d,$$

and similarly
$$u_{m-1} = c - d.$$

So $(u_{k-1}, u_k) = (u_{m-1}, u_m) = (c - d, d)$. But since $k \geq 1$, $2k - 1 > k - 1$, and this implies that u_{2k-1} and u_{2k} is not the first pair of consecutive integers appearing twice, hence leading to a contradiction.

Both cases lead to a contradiction, so it is not possible for two positive integers to appear consecutively in the same order in two different places in the sequence, as desired.

4. In the case where $a > b$, if a and b do not occur consecutively with b following a, then there does not exist a $k \geq 1$ such that
$$u_{2k-1} = a, u_{2k} = b.$$

If there exists $m \geq 2$ such that
$$u_{m-1} = a - b, u_m = b,$$

then notice
$$u_{2m-1} = u_{m-1} + u_m = a, u_{2m} = u_m = b,$$

and for $k = m$ we have $u_{2k-1} = a$ and $u_{2k} = b$. Hence, such m does not exist, and $a - b$ and b are two co-prime positive integers which do not occur consecutively in the sequence with b following $a - b$, and whose sum is smaller than $a + b$.

Similarly, in the case where $a < b$, if a and b do not occur consecutively with b following a, then there does not exist a $k \geq 1$ such that

$$u_{2k} = a, u_{2k+1} = b.$$

If there exists $m \geq 1$ such that
$$u_m = a, u_{m+1} = b - a,$$

then notice
$$u_{2m} = u_m = a, u_{2m+1} = u_m + u_{m+1} = b,$$

and for $k = m$ we have $u_{2k} = a$ and $u_{2k+1} = b$. Hence, such m does not exist, and a and $b - a$ are two co-prime positive integers which do not occur consecutively in the sequence with $b-a$ following a, and whose sum is smaller than $a + b$.

5. Suppose that there is some rational number $q = \frac{a}{b}$ where $\gcd(a, b) = 1$, $a, b > 0$ which is not in the range of f. Let a, b be such that the sum $a + b$ is the lowest. Then, there does not exist an integer $n \geq 1$, such that
$$f(n) = \frac{a}{b}.$$

Since all consecutive terms in the sequence are co-prime, this means there does not exist an integer $n \geq 1$, such that

$$u_n = a, u_{n+1} = b.$$

If $a > b$, then the pair $(a - b, b)$ with a sum less than $a + b$ must not exist consecutively in the sequence either, which contradicts with that $a + b$ is the pair with the smallest sum.

If $a < b$, then the pair $(b, b - a)$ with a sum less than $a + b$ must not exist consecutively in the sequence either, which contradicts with that $a + b$ is the pair with the smallest sum.

If $a = b$, then the only possibility is $a = b = 1$, but $n = 1$ gives $u_1 = 1$ and $u_2 = 1$, so this is not possible.

Hence, such rational number which is not within the range doesn't exist, and the range of f is all positive rational numbers.

Since the fraction representation of a positive rational number is unique (given the numerator and denominator are co-prime and both positive), and all terms in the sequence are positive, consecutive terms are co-prime, and consecutive terms do not appear again in this order, it must be that case that there is at most one pair of consecutive terms that gives the ratio of any positive rational number q, which shows that f has an inverse.

Hence, f has a range of all positive rational numbers, and f has an inverse, as desired.

2020.3 Question 11

1. Since $X \sim \mathrm{U}[a, b]$, we must have for the probability density function of X, that

$$f_X(x) = \frac{1}{b-a}$$

for $x \in [a, b]$, and 0 everywhere else. Hence, the cumulative distribution function of X is

$$F_X(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{b-a}, & a < x \leq b, \\ b, & \text{otherwise.} \end{cases}$$

Since f is bijective and strictly decreasing on $[a, b]$, we must have for $y \in [a, b]$, that

$$\begin{aligned} \mathrm{P}(Y \leq y) &= \mathrm{P}(f(X) \leq y) \\ &= \mathrm{P}(X \geq f^{-1}(y)) \\ &= \mathrm{P}(X \geq f(y)) \\ &= 1 - \mathrm{P}(X < f(y)) \\ &= 1 - F_X(f(y)) \\ &= 1 - \frac{f(y) - a}{b - a} \\ &= \frac{(b-a) - (f(y) - a)}{b - a} \\ &= \frac{b - f(y)}{b - a}, \end{aligned}$$

as desired.

Hence, by differentiation with respect to y, we have the probability density function of Y satisfies

$$f_Y(y) = -\frac{f'(y)}{b-a}.$$

Hence, by the definition of expectation, we have

$$\begin{aligned} \mathrm{E}(y^2) &= \int_a^b f_Y(y) y^2 \, \mathrm{d}y \\ &= -\frac{1}{b-a} \int_a^b -f'(y) y^2 \, \mathrm{d}y \\ &= -\frac{1}{b-a} \int_a^b y^2 \, \mathrm{d}f(y) \\ &= \frac{1}{b-a} \left[- \left[y^2 f(y) \right]_a^b - 2 \int_a^b y f(y) \, \mathrm{d}Y \right] \\ &= \frac{1}{b-a} \left[-b^2 f(b) + a^2 f(a) + 2 \int_a^b y f(y) \, \mathrm{d}Y \right] \\ &= \frac{1}{b-a} \left[\frac{b}{3}(b^3 - a^3) - b^2 a + a^2 b + 2 \int_a^b y f(y) \, \mathrm{d}x \right] \\ &= \frac{b}{3} \left(b^2 + ab + a^2 \right) - ab + \int_a^b \frac{2x f(x) \, \mathrm{d}x}{b - a}. \end{aligned}$$

2. Since $\frac{1}{Z} + \frac{1}{X} = \frac{1}{c}$, by rearranging, we have

$$Z = \frac{1}{\frac{1}{c} - \frac{1}{X}} = \frac{cX}{X - c}.$$

By given, we have

$$c = \frac{ab}{a+b},$$

and hence

$$c < a, c < b.$$

Let $f(x) = \frac{cx}{x-c}$. Notice that

$$\begin{aligned} f(a) &= \frac{ac}{a-c} \\ &= \frac{a^2b/(a+b)}{a-ab/(a+b)} \\ &= \frac{a^2b}{a^2+ab-ab} \\ &= b, \end{aligned}$$

and

$$\begin{aligned} f(b) &= \frac{bc}{b-c} \\ &= \frac{ab^2/(a+b)}{b-ab/(a+b)} \\ &= \frac{ab^2}{b^2+ab-ab} \\ &= a. \end{aligned}$$

Also, since

$$f(x) = \frac{1}{\frac{x-c}{cx}} = \frac{1}{\frac{1}{c} - \frac{1}{x}},$$

as x strictly increases, $\frac{1}{x}$ strictly decreases, $-\frac{1}{x}$ strictly increases, the denominator strictly increases, and hence $f(x)$ strictly decreases.

Note that

$$\frac{1}{f(x)} + \frac{1}{x} = \frac{1}{c},$$

and hence

$$\frac{1}{x} + \frac{1}{f^{-1}(x)} = \frac{1}{c},$$

which implies

$$f(x) = f^{-1}(x).$$

So $Z = f(X)$ for this f satisfying all three conditions above. Hence,

$$\begin{aligned} \mathrm{E}(Z) &= \int_a^b f(x) f_X(x)\,\mathrm{d}x \\ &= \frac{1}{b-a}\int_a^b \frac{cx\,\mathrm{d}x}{x-c} \\ &= \frac{1}{b-a}\int_a^b \left(c + \frac{c^2}{x-c}\right)\mathrm{d}x \\ &= \frac{1}{b-a}\left[cx + c^2\ln|x-c|\right]_a^b \\ &= \frac{1}{b-a}\left[\left(cb + c^2\ln|b-c|\right) - \left(ca + c^2\ln|a-c|\right)\right] \\ &= c + \frac{c^2}{b-a}\ln\left|\frac{b-c}{a-c}\right| \\ &= c + \frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right), \end{aligned}$$

and using the result from the previous part,

$$\begin{aligned}
\mathrm{E}(Z^2) &= -ab + \int_a^b \frac{2xf(x)}{b-a}\,\mathrm{d}x \\
&= -ab + \frac{2}{b-a} \cdot \int_a^b \frac{cx^2}{x-c}\,\mathrm{d}x \\
&= -ab + \frac{2c}{b-a} \cdot \int_a^b \left(x + c + \frac{c^2}{x-c}\right)\mathrm{d}x \\
&= -ab + \frac{2c}{b-a} \cdot \left[\frac{x^2}{2} + cx + c^2 \ln|x-c|\right]_a^b \\
&= -ab + \frac{2c}{b-a} \cdot \left[\left(\frac{b^2}{2} + bc + c^2 \ln|b-c|\right) - \left(\frac{a^2}{2} + ac + c^2 \ln|a-c|\right)\right] \\
&= -ab + \frac{2c}{b-a} \cdot \left[(b-a)\left(c + \frac{a+b}{2}\right) + c^2 \ln\left|\frac{b-c}{a-c}\right|\right] \\
&= -ab + 2c\left(c + \frac{a+b}{2}\right) + \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right) \\
&= 2c^2 + (a+b)c - ab + \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right) \\
&= 2c^2 + \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right).
\end{aligned}$$

Hence, the variance of Z satisfies that

$$\begin{aligned}
\mathrm{Var}(Z) &= \mathrm{E}(Z^2) - \mathrm{E}(Z)^2 \\
&= 2c^2 + \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right) - \left(c + \frac{c^2}{b-a} \ln\left(\frac{b-c}{a-c}\right)\right) \\
&= 2c^2 + \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right) - c^2 - \frac{2c^3}{b-a} \ln\left(\frac{b-c}{a-c}\right) - \frac{c^4}{(b-a)^2} \ln\left(\frac{b-c}{a-c}\right)^2 \\
&= c^2 - \frac{c^4}{(b-a)^2} \ln\left(\frac{b-c}{a-c}\right)^2.
\end{aligned}$$

Therefore, since the variance of a non-constant random variable is always positive,

$$\begin{aligned}
c^2 &> \frac{c^4}{(b-a)^2} \ln\left(\frac{b-c}{a-c}\right)^2 \\
(b-a)^2 &> c^2 \ln\left(\frac{b-c}{a-c}\right)^2 \\
|b-a| &> \left|c \ln\left(\frac{b-c}{a-c}\right)\right| \\
\left|\ln\left(\frac{b-c}{a-c}\right)\right| &< \left|\frac{b-a}{c}\right|.
\end{aligned}$$

Notice that since $b > a$, we must have $b - c > a - c$, so the natural log on the left-hand side is positive, and the fraction within the absolute value on the right-hand side is positive as well, and hence

$$\ln\left(\frac{b-c}{a-c}\right) < \frac{b-a}{c}.$$

2020.3 Question 12

1. By the definition within the question, we have that $X, Y \sim \text{Geo}(p)$, and for $t \geq 1$,

$$\text{P}(X = t) = \text{P}(Y = t) = q^{t-1}p.$$

For $S = X + Y$, we have for $s \geq 2$,

$$\begin{aligned}
\text{P}(S = s) &= \text{P}(X + Y = s) \\
&= \sum_{t=1}^{s-1} \text{P}(X = t, Y = s - t) \\
&= \sum_{t=1}^{s-1} \text{P}(X = t)\,\text{P}(Y = s - t) \\
&= \sum_{t=1}^{s-1} q^{t-1}pq^{s-t-1}p \\
&= \sum_{t=1}^{s-1} q^{s-2}p^2 \\
&= (s-1)q^{s-2}p^2.
\end{aligned}$$

For $T = \max\{X, Y\}$, we have for $t \geq 1$,

$$\begin{aligned}
\text{P}(T = t) &= \text{P}(X = Y = t) + \text{P}(X = t, Y < X) + \text{P}(Y = t, X < Y) \\
&= \text{P}(X = t, Y = t) + 2\,\text{P}(X = t, Y < X) \\
&= \text{P}(X = t)\,\text{P}(Y = t) + 2\sum_{r=1}^{t-1} \text{P}(X = t, Y = r) \\
&= \text{P}(X = t)\,\text{P}(Y = t) + 2\sum_{r=1}^{t-1} \text{P}(X = t)\,\text{P}(Y = r) \\
&= q^{t-1}pq^{t-1}p + 2q^{t-1}p\sum_{r=1}^{t-1} q^{r-1}p \\
&= q^{2t-2}p^2 + 2q^{t-1}p^2\sum_{r=1}^{t-1} q^{r-1} \\
&= q^{2t-2}p^2 + 2q^{t-1}p^2\frac{1 - q^{t-1}}{1 - q} \\
&= q^{2t-2}p^2 + 2q^{t-1}p^2\frac{1 - q^{t-1}}{p} \\
&= q^{2t-2}p^2 + 2q^{t-1}p(1 - q^{t-1}) \\
&= pq^{t-1}\left(pq^{t-1} + 2 - 2q^{t-1}\right) \\
&= pq^{t-1}\left((1-q)q^{t-1} + 2 - 2q^{t-1}\right) \\
&= pq^{t-1}\left(2 + q^t - q^{t-1}\right)
\end{aligned}$$

2. Since $U = |X - Y|$, we have $U \geq 0$. For $u \geq 1$, we have

$$\begin{aligned}
\mathrm{P}(U = u) &= \mathrm{P}(|X - Y| = u) \\
&= \mathrm{P}(X - Y = \pm u) \\
&= 2\,\mathrm{P}(X - Y = u) \\
&= 2\sum_{t=1}^{\infty} \mathrm{P}(X = u + t, Y = t) \\
&= 2\sum_{t=1}^{\infty} \mathrm{P}(X = u + t)\,\mathrm{P}(Y = t) \\
&= 2\sum_{t=1}^{\infty} q^{u+t-1} p q^{t-1} p \\
&= 2q^u p^2 \sum_{t=1}^{\infty} q^{2t-2} \\
&= 2q^u p^2 \cdot \frac{1}{1 - q^2} \\
&= 2q^u p^2 \cdot \frac{1}{(1 + q)p} \\
&= \frac{2q^u p}{1 + q},
\end{aligned}$$

and for $u = 0$,

$$\begin{aligned}
\mathrm{P}(U = 0) &= \mathrm{P}(X = Y) \\
&= \sum_{t=1}^{\infty} \mathrm{P}(X = Y = t) \\
&= \sum_{t=1}^{\infty} \mathrm{P}(X = t)\,\mathrm{P}(Y = t) \\
&= \sum_{t=1}^{\infty} q^{t-1} p q^{t-1} p \\
&= p^2 \sum_{t=1}^{\infty} q^{2t-2} \\
&= p^2 \cdot \frac{1}{1 - q^2} \\
&= \frac{p}{1 + q}.
\end{aligned}$$

Since $W = \min\{X, Y\}$, we have $W \geq 1$. For $w \geq 1$, we have

$$\begin{aligned}
\mathrm{P}(W = w) &= \mathrm{P}(X = Y = w) + \mathrm{P}(X = w, Y > X) + \mathrm{P}(Y = w, Y < X) \\
&= \mathrm{P}(X = w, Y = w) + 2\,\mathrm{P}(X = w, Y > X) \\
&= \mathrm{P}(X = w)\,\mathrm{P}(Y = w) + 2\sum_{r=w+1}^{\infty} \mathrm{P}(X = w, Y = r) \\
&= \mathrm{P}(X = w)\,\mathrm{P}(Y = w) + 2\sum_{r=w+1}^{\infty} \mathrm{P}(X = w)\,\mathrm{P}(Y = r) \\
&= q^{w-1}pq^{w-1}p + 2\sum_{r=w+1}^{\infty} q^{w-1}pq^{r-1}p \\
&= q^{2w-2}p^2 + 2q^{w-2}p^2 \sum_{r=w+1}^{\infty} q^r \\
&= q^{2w-2}p^2 + 2q^{w-2}p^2q^{w+1} \cdot \frac{1}{1-q} \\
&= q^{2w-2}p^2 + 2q^{2w-1}p^2 \cdot \frac{1}{p} \\
&= q^{2w-2}p^2 + 2q^{2w-1}p \\
&= q^{2w-2}p\,(p + 2q) \\
&= q^{2w-2}p\,(1 + q)\,.
\end{aligned}$$

3. Since $S = 2$ and $T = 3$, the maximum of X and Y is 3, but they sum to 2, and this is impossible, so

$$\mathrm{P}(S = 2, T = 3) = 0.$$

However,

$$\begin{aligned}
\mathrm{P}(S = 2)\,\mathrm{P}(T = 3) &= (2-1)q^{2-2}p^2 \cdot pq^{3-1}(2 + q^3 - q^{3-1}) \\
&= p^3q^2(2 + q^3 - q^2) \\
&\neq 0 \\
&= \mathrm{P}(S = 2, T = 3),
\end{aligned}$$

as desired.

4. • U and W are independent. We split this into two cases of U to consider:
   - When $U = 0$, $X = Y$, and hence $W = X = Y$. In this case,

     $$\mathrm{P}(U = 0, W = w) = \mathrm{P}(X = Y = w) = q^{2w-2}p^2$$

     and notice

     $$\mathrm{P}(U = 0)\,\mathrm{P}(W = w) = \frac{p}{1+q} \cdot q^{2w-2}p(1+q) = p^2q^{2w-2},$$

     so

     $$\mathrm{P}(U = 0, W = w) = \mathrm{P}(U = 0)\,\mathrm{P}(W = w).$$

   - When $U = u \neq 0$,

     $$\begin{aligned}
     \mathrm{P}(U = u, W = w) &= \mathrm{P}(X = w, Y = w + u) + \mathrm{P}(X = w + u, Y = w) \\
     &= 2\,\mathrm{P}(X = w, Y = w + u) \\
     &= 2\,\mathrm{P}(X = w)\,\mathrm{P}(Y = w + u) \\
     &= 2q^{w-1}pq^{w+u-1}p \\
     &= 2q^{2w+u-2}p^2,
     \end{aligned}$$

     and

     $$\mathrm{P}(U = u)\,\mathrm{P}(W = w) = \frac{2q^u p}{1+q} \cdot q^{2w-2}p(1+q) = 2q^{2w+u-2}p^2,$$

     and so

     $$\mathrm{P}(U = u, W = w) = \mathrm{P}(U = u)\,\mathrm{P}(W = w).$$

Hence, we can see that U and W are independent.

- U and S are not independent. Consider the case where $S = 3$ and $U = 0$. The event $S = 3, U = 0$ is not possible since $S = X + Y$ and $U = |X - Y|$ must have the same odd-even parity, giving

$$\mathrm{P}(S = 3, U = 0) = 0.$$

On the other hand,

$$\mathrm{P}(S = 3)\,\mathrm{P}(U = 0) = 2qp^2 \cdot \frac{p}{1+q} = \frac{2qp^3}{1+p} \neq 0.$$

This means

$$\mathrm{P}(S = 3, U = 0) \neq \mathrm{P}(S = 3)\,\mathrm{P}(U = 0),$$

and hence U and S are not independent.

- U and T are not independent. Consider the case where $U = 1$ and $T = 1$. The event $U = 1, T = 1$ implies that $X = Y \pm 1$, and that the maximum of X and Y is 1, and hence $X = Y = 1$, which is impossible. Hence,

$$\mathrm{P}(U = 1, T = 1) = 0.$$

On the other hand,

$$\mathrm{P}(U = 1)\,\mathrm{P}(T = 1) = \frac{2qp}{1+q} \cdot p(2 + q - 1) = 2p^2q \neq 0.$$

This means

$$\mathrm{P}(U = 1, T = 1) \neq \mathrm{P}(U = 1)\,\mathrm{P}(T = 1),$$

and hence U and T are not independent.

- W and S are not independent. Consider the case where $W = 2$ and $S = 2$. On one hand, since $\min\{X, Y\} = 2$, and $S = X + Y = \max\{X, Y\} + \min\{X, Y\} = 2$, this means $\max\{X, Y\} = 0$ which is impossible, and hence

$$\mathrm{P}(W = 2, S = 2) = 0.$$

On the other hand,

$$\mathrm{P}(W = 2)\,\mathrm{P}(S = 2) = q^2p(1 + q)(1)q^0p^2 = p^3q^2(1 + q) \neq 0.$$

This means

$$\mathrm{P}(W = 2, S = 2) = \mathrm{P}(W = 2)\,\mathrm{P}(S = 2).$$

- W and T are not independent. Consider the case where $T = 1$ and $W = 2$. Since $T = \max\{X, Y\} = 1$ and $W = \min\{X, Y\} = 2$, the event $T = 1, W = 2$ is not possible, hence

$$\mathrm{P}(T = 1, W = 2) = 0.$$

On the other hand,

$$\mathrm{P}(T = 1)\,\mathrm{P}(W = 2) = p(2 + q - 1)q^2p(1 + q) = p^2q^2(1 + q)^2 \neq 0.$$

This means

$$\mathrm{P}(T = 1, W = 2) \neq \mathrm{P}(T = 1)\,\mathrm{P}(W = 2),$$

and hence W and T are not independent.

- S and T are not independent. Counter-example shown in the previous part.