2018 Paper 3

2018.3 Question 1

1. By differentiation with respect to β, we have

$$f'(\beta) = 1 + \frac{1}{\beta^2} + \frac{2}{\beta^3}.$$

If $f'(t) = 0$, we must have

$$t^3 + t + 2 = 0.$$

Therefore,

$$(t+1)(t^2 - t + 2) = 0,$$

and hence the only real root to this is $t = -1$, since $(-1)^2 - 2 \cdot 4 < 0$.

This means the only stationary point of $y = f(\beta)$ is $(-1, f(-1) = -1)$.

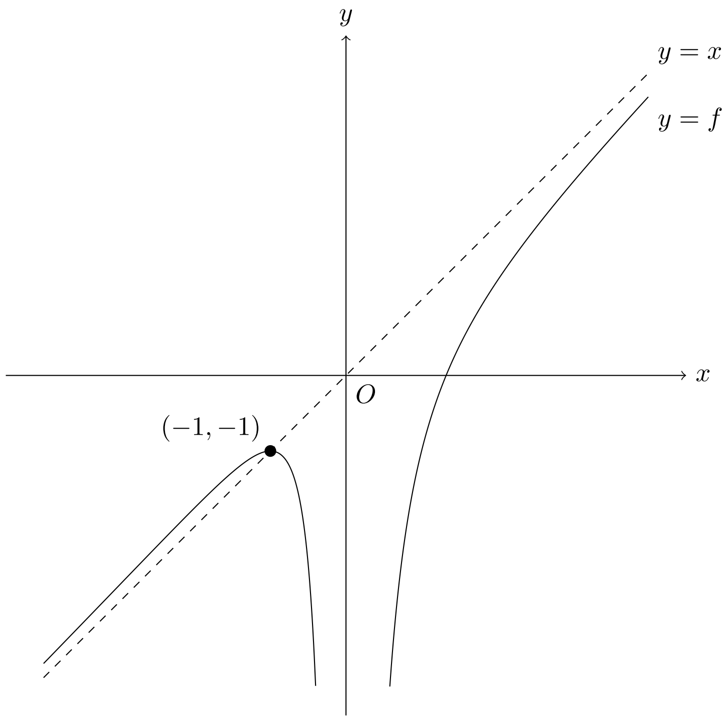
For the limiting behaviour of the function, we first look at the case where $\beta > 0$. As $\beta \to \infty$, we have $f(\beta) \to \beta$ from below. As $\beta \to 0^+$, we have $f(\beta) \to -\frac{1}{\beta} - \frac{1}{\beta^2} \to -\infty$.

When $\beta < 0$, we use the substitution $t = -\frac{1}{\beta}$ to make the behaviours more convincing, and hence

$$f(\beta) = \beta + t - t^2.$$

As $\beta \to 0^-$, we have $t \to \infty$, and $f(\beta) \to t - t^2 \to -\infty$. As $\beta \to -\infty$, we have $t \to 0^+$, and $f(\beta) \to \beta$ from above, since $t - t^2 = t(1-t) > 0$ when $0 < t < 1$.

This means the curve $y = f(\beta)$ is as below.



Similarly, by differentiation with respect to β, we have

$$g'(\beta) = 1 - \frac{3}{\beta^2} + \frac{2}{\beta^3}.$$

If $g'(t) = 0$, we must have

$$t^3 - 3t + 2 = 0.$$

Therefore,

$$(t-1)^2(t+2) = 0,$$

and hence the real roots to this is $t = 1$ and $t = -2$.

This means the stationary points of $y = g(\beta)$ is $(1, g(1) = 3)$ and $(-2, g(-2) = -\frac{15}{4})$.

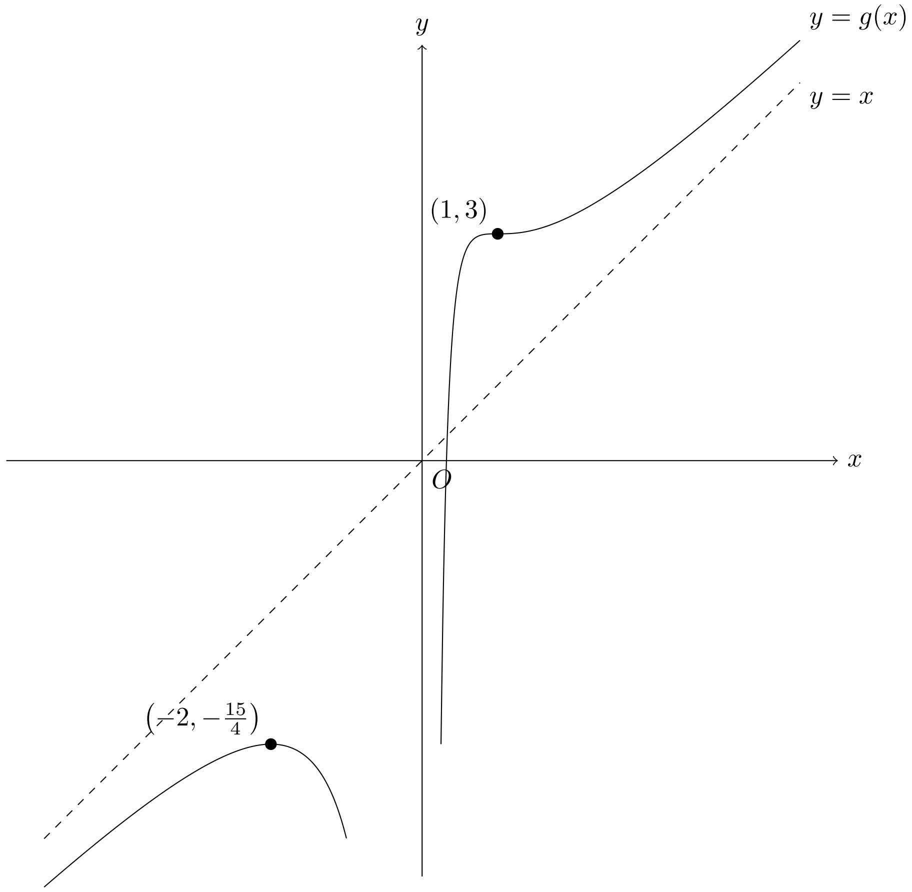
For the limiting behaviour of the function, we first look at the case where $\beta > 0$. We consider the substitution $t = -\frac{1}{\beta}$ to make the behaviours more convincing, and hence

$$g(\beta) = \beta - 3t - t^2.$$

As $\beta \to \infty$, $t \to 0^-$, and hence $f(\beta) \to \beta$ from below, since $-3t - t^2 = -t(t+3) > 0$ for $-3 < t < 0$. As $\beta \to 0^+$, $t \to -\infty$, and hence $f(\beta) \to -3t - t^2 \to -\infty$.

When $\beta < 0$, we have as $\beta \to 0^-$, $f(\beta) \to -\infty$. As $\beta \to -\infty$, $f(\beta) \to \beta$ from below.

This means the curve $y = g(\beta)$ is as below.



2. By Vieta's Theorem, we have $u + v = -\alpha$, and $uv = \beta$. Hence,

$$u + v + \frac{1}{uv} = -\alpha + \frac{1}{\beta},$$

and

$$\frac{1}{u} + \frac{1}{v} + uv = \frac{u+v}{uv} + uv = -\frac{\alpha}{\beta} + \beta.$$

3. By the given condition, we have

$$-\alpha + \frac{1}{\beta} = -1 \iff \alpha = 1 + \frac{1}{\beta}.$$

Hence,

$$\begin{aligned}\frac{1}{u}+\frac{1}{v}+uv&=-\frac{\alpha}{\beta}+\beta\\&=-\frac{1+\frac{1}{\beta}}{\beta}+\beta\\&=\frac{\beta^2-1-\frac{1}{\beta}}{\beta}\\&=\beta-\frac{1}{\beta}-\frac{1}{\beta^2}\\&=f(\beta).\end{aligned}$$

Also, since u, v are both real, we have

$$\begin{aligned}\alpha^2-4\beta&=\left(1+\frac{1}{\beta}\right)^2-4\beta\\&=1+\frac{2}{\beta}+\frac{1}{\beta^2}-4\beta\\&=\frac{-4\beta^3+\beta^2+2\beta+1}{\beta^2}\\&\geq 0.\end{aligned}$$

Multiplying both sides by $-\beta^2$ (which flips the sign) gives

$$\begin{aligned}4\beta^3-\beta^2-2\beta-1\leq 0\\(\beta-1)(4\beta^2+3\beta+1)\leq 0.\end{aligned}$$

This cubic has exactly one real root $\beta = 1$, so the solution to this inequality is $\beta \leq 1$ and $\beta \neq 0$. Notice that f is increasing on $(0, 1] \subset (0, \infty)$. Therefore, for $\beta > 0$,

$$f(\beta) \leq f(1) = 1 - 1 - 1 = -1.$$

When $\beta < 0$, we have

$$f(\beta) \leq f(-1) = -1.$$

So for the range of β in this question, we always have $f(\beta) \leq -1$. But we also have $\frac{1}{u}+\frac{1}{v}+uv \leq -1$ as shown before. These gives us exactly our desired statement.

4. By the given condition, we have

$$-\alpha+\frac{1}{\beta}=3 \iff \alpha=-3+\frac{1}{\beta}.$$

Hence,

$$\begin{aligned}\frac{1}{u}+\frac{1}{v}+uv&=-\frac{\alpha}{\beta}+\beta\\&=-\frac{-3+\frac{1}{\beta}}{\beta}+\beta\\&=\beta+\frac{3}{\beta}-\frac{1}{\beta^2}\\&=g(\beta).\end{aligned}$$

Also, since u, v are both real, we have $\beta \leq 1$ and $\beta \neq 0$ as well.

g must be increasing on $(0, 1]$. Hence, for $\beta > 0$, we have

$$g(\beta) \leq g(1) = 3.$$

When $\beta < 0$, we have

$$g(\beta) \leq g(-2) = -\frac{15}{4}.$$

Since $3 > -\frac{15}{4}$, we can conclude that the maximum value of $\frac{1}{u} + \frac{1}{v} + uv$ is 3, and it is taken when $\beta = 1$, which corresponds to $\alpha = -2$.

2018.3 Question 2

1. Notice that

$$\begin{aligned}
\frac{\mathrm{d}y_n}{\mathrm{d}x} &= \frac{\mathrm{d}(-1)^n \frac{1}{z}\frac{\mathrm{d}^n z}{\mathrm{d}x^n}}{\mathrm{d}x} \\
&= (-1)^n \left[\frac{\mathrm{d}\frac{1}{z}}{\mathrm{d}x} \cdot \frac{\mathrm{d}^n z}{\mathrm{d}x^n} + \frac{1}{z} \cdot \frac{\mathrm{d}\frac{\mathrm{d}^n z}{\mathrm{d}x^n}}{\mathrm{d}x}\right] \\
&= (-1)^n \left[\frac{2x}{z} \cdot \frac{\mathrm{d}^n z}{\mathrm{d}x^n} + \frac{1}{z} \cdot \frac{\mathrm{d}^{n+1} z}{\mathrm{d}x^{n+1}}\right] \\
&= 2x \cdot (-1)^n \frac{1}{z}\frac{\mathrm{d}^n z}{\mathrm{d}x^n} - (-1)^{n+1}\frac{1}{z}\frac{\mathrm{d}^{n+1} z}{\mathrm{d}x^{n+1}} \\
&= 2xy_n - y_{n+1},
\end{aligned}$$

as desired.

2. We first look at the base case where $n = 1$. What is desired is

$$y_2 = 2xy_1 - 2y_0.$$

We have $y_0 = 1$,

$$y_1 = (-1)^1 \frac{1}{e^{-x^2}} \frac{\mathrm{d}e^{-x^2}}{\mathrm{d}x} = -e^{x^2}(-2x)e^{-x^2} = 2x,$$

and

$$y_2 = 2xy_1 - \frac{\mathrm{d}y_1}{\mathrm{d}x} = 2x \cdot 2x - 2 = 4x^2 - 2.$$

Hence,

$$2xy_1 - 2y_0 = 2x \cdot 2x - 2 \cdot 1 - 4x^2 - 2 = y_2,$$

so the base case is satisfied.

Now assume this is true for some $n = k \geq 1$, i.e.

$$y_{k+1} = 2xy_k - 2ky_{k-1}.$$

We have

$$\begin{aligned}
y_{k+2} &= 2xy_{k+1} - \frac{\mathrm{d}y_{k+1}}{\mathrm{d}x} \\
&= 2xy_{k+1} - \frac{\mathrm{d}\left(2xy_k - 2ky_{k-1}\right)}{\mathrm{d}x} \\
&= 2xy_{k+1} - 2y_k - 2x\frac{\mathrm{d}y_k}{\mathrm{d}x} + 2k\frac{\mathrm{d}y_{k-1}}{\mathrm{d}x} \\
&= 2xy_{k+1} - 2y_k - 2x(2xy_k - y_{k+1}) + 2k(2xy_{k-1} - y_k) \\
&= 2xy_{k+1} - 2y_k - 4x^2y_k + 2xy_{k+1} + 4kxy_{k-1} - 2ky_k \\
&= 4xy_{k+1} - 2(2x^2 + k + 1)y_k + 4kxy_{k-1} \\
&= 4xy_{k+1} - 2(2x^2 + k + 1)y_k + 4kx \cdot \frac{2xy_k - y_{k+1}}{2k} \\
&= 4xy_{k+1} - 2(2x^2 + k + 1)y_k + 2x(2xy_k - y_{k+1}) \\
&= 2xy_{k+1} - 2(k + 1)y_k,
\end{aligned}$$

which is exactly the statement for $n = k + 1$.

Hence, by the principle of mathematical induction, we have $y_{n+1} = 2xy_n - 2ny_{n-1}$ for all $n \geq 1$.

We have

$$\begin{aligned}
\text{LHS} &= y_{n+1}^2 - y_n y_{n+2} \\
&= y_{n+1}^2 - y_n(2xy_{n+1} - 2(n+1)y_n) \\
&= y_{n+1}^2 - 2xy_n y_{n+1} + 2(n+1)y_n^2
\end{aligned}$$

and

$$\begin{aligned}\text{RHS} &= 2n(y_n^2 - y_{n-1}y_{n+1}) + 2y_n^2 \\ &= 2n\left(y_n^2 - \frac{2xy_n - y_{n+1}}{2n}y_{n+1}\right) + 2y_n^2 \\ &= 2ny_n^2 - (2xy_n - y_{n+1})\,y_{n+1} + 2y_n^2 \\ &= 2ny_n^2 - 2xy_ny_{n+1} + y_{n+1}^2 + 2y_n^2 \\ &= y_{n+1}^2 - 2xy_ny_{n+1} + 2(n+1)y_n^2.\end{aligned}$$

3. This can be shown by induction on n. The base case for $n = 1$ is

$$y_1^2 - y_0y_2 = (2x)^2 - 1 \cdot (4x^2 - 2) = 2 > 0$$

is true.

Now assume the statement is true for $n = k \geq 1$, i.e.

$$y_k^2 - y_{k-1}y_{k+1} > 0.$$

We have

$$\begin{aligned}y_{k+1}^2 - y_ky_{k+2} &= 2n(y_k^2 - y_{k-1}y_k + 1) + 2y_n^2 \\ &> 2n \cdot 0 + y_n^2 \\ &= 0 + y_n^2 \\ &\geq 0,\end{aligned}$$

which is the statement for $n = k + 1$.

Hence, by the principle of mathematical induction, we have $y_n^2 - y_{n-1}y_{n+1} > 0$ for all $n \geq 1$.

2018.3 Question 3

Notice that

$$\begin{aligned}x^a(x^b(x^c y)')' &= x^a(x^b(cx^{c-1}y + x^c y'))' \\ &= x^a\left[x^{b+c-1}\left(cy + xy'\right)\right]' \\ &= x^a\left[(b+c-1)x^{b+c-2}\left(cy+xy'\right) + x^{b+c-1}\left(cy' + y' + xy''\right)\right] \\ &= x^{a+b+c-2}\left[(b+c-1)\left(cy+xy'\right) + x\left(cy'+y'+xy''\right)\right] \\ &= x^{a+b+c-2}\left[x^2y'' + (b+2c)xy' + (b+c-1)cy\right].\end{aligned}$$

Comparing this with the left-hand side of the original equation, we must have

$$\begin{cases} a+b+c-2=0, \\ b+2c = 1-2p, \\ (b+c-1)c = p^2-q^2. \end{cases}$$

The second equation gives

$$b = 1-2p-2c,$$

and putting this into the third equation gives

$$(1-2p-2c+c-1)c = p^2-q^2,$$

and hence

$$c^2+2pc+p^2-q^2=0.$$

This gives

$$(c+(p-q))(c+(p+q))=0,$$

and hence

$$c_1 = -p+q, c_2 = -p-q.$$

Putting this back, we get

$$b_1 = 1-2p-2(-p+q) = 1-2q, b_2 = 1-2p-2(-p-q) = 1+2q,$$

and since $a = 2-b-c$ from the first equation, we have

$$a_1 = 2-(1-2q)-(-p+q) = 1+p+q$$

and

$$a_2 = 2-(1+2q)-(-p-q) = 1+p-q$$

Hence, the solutions are

$$\begin{cases} a = p \pm q + 1, \\ b = \mp 2q + 1, \\ c = -p \pm q. \end{cases}$$

1. In the case where $f(x)=0$. We must have

   $$x^a\left(x^b(x^c y)'\right)' = 0,$$

   and hence

   $$\left(x^b(x^c y)'\right)' = 0.$$

   Therefore, we must have by integration

   $$x^b(x^c y)' = C_1$$

   for some (real) constant C_1.
   Hence,

   $$(x^c y)' = C_1 x^{-b}.$$

   There are two cases here:

(a) When $b = 1$ i.e. $q = 0$, the right-hand side is $C_1 x^{-1}$, and the left-hand side is $(x^c y)'$. Integrating both sides give
$$x^c y = C_1 \ln x + C_2$$
for some (real) constant C_2.
Hence,
$$y = x^{-c}(C_1 \ln x + C_2)$$
for some (real) constants C_1, C_2.
When $q = 0$, $c = -p$, and hence
$$y = x^p(C_1 \ln x + C_2).$$

(b) When $b \neq 1$ i.e. $q \neq 0$, integrating both sides give
$$x^c y = \frac{C_1 x^{-b+1}}{-b+1} + C_2$$
for some (real) constant C_2.
Hence,
$$y = x^{-c}\left(\frac{C_1 x^{-b+1}}{-b+1} + C_2\right)$$
for some (real) constant C_1, C_2.
Hence,
$$\begin{aligned} y &= x^{-(-p \pm q)}\left(\frac{C_1 x^{-(\mp 2q+1)+1}}{-(\mp 2q+1)+1} + C_2\right) \\ &= x^{p \mp q}\left(\frac{C_1 x^{\pm 2q}}{\pm 2q} + C_2\right). \\ &= \frac{C_1}{\pm 2q} x^{p \pm q} + C_2 x^{p \mp q} \\ &= C_3 x^{p \pm q} + C_2 x^{p \mp q}, \end{aligned}$$
for some (real) constant C_2, C_3.

2. This is when $q = 0$ and $f(x) = x^n$. We have $a = p + 1, b = 1$ and $c = -p$, and the original differential equation reduces to
$$x^{p+1}\left(x\left(x^{-p}y\right)'\right)' = x^n,$$
and hence
$$\left(x\left(x^{-p}y\right)'\right)' = x^{n-p-1}.$$
There are two cases here:

(a) If $n - p - 1 = -1$, i.e. $n = p$, we have, by integration,
$$x\left(x^{-p}y\right)' = \ln x + C_1.$$
This gives
$$\left(x^{-p}y\right)' = \frac{\ln x}{x} + \frac{C_1}{x},$$
and hence by integration
$$x^{-p}y = \frac{(\ln x)^2}{2} + C_1 \ln x + C_2.$$
This solves to
$$y = \frac{x^p(\ln x)^2}{2} + C_1 x^p \ln x + C_2 x^p.$$

(b) If $n - p - 1 \neq -1$, i.e. $n \neq p$, we have

$$x \left(x^{-p} y\right)' = \frac{x^{n-p}}{n-p} + C_1.$$

This gives

$$\left(x^{-p} y\right)' = \frac{x^{n-p-1}}{n-p} + \frac{C_1}{x}.$$

Since $n - p - 1 \neq -1$, by integration we have

$$x^{-p} y = \frac{x^{n-p}}{(n-p)^2} + C_1 \ln x + C_2,$$

and hence

$$y = \frac{x^n}{(n-p)^2} + C_1 x^p \ln x + C_2 x^p.$$

2018.3 Question 4

The hyperbola has parametric equation

$$\begin{cases} x = a\sec\theta, \\ y = b\tan\theta. \end{cases}$$

Hence, by differentiation, we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} \\ &= \frac{b\sec^2\theta}{a\sec\theta\tan\theta} \\ &= \frac{b\cos\theta}{a\sin\theta\cos\theta} \\ &= \frac{b}{a\sin\theta}.\end{aligned}$$

The tangent to the hyperbola at P will be

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta),$$

which simplifies to

$$ay\sin\theta - ab\tan\theta\sin\theta = bx - ab\sec\theta,$$

and hence

$$bx - ay\sin\theta = ab(\sec\theta - \tan\theta\sin\theta).$$

Notice that

$$\sec\theta - \tan\theta\sin\theta = \frac{1-\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta} = \cos\theta,$$

and so the equation of the tangent is

$$bx - ay\sin\theta = ab\cos\theta,$$

exactly as desired.

1. Let $\frac{x}{a} = \frac{y}{b} = s$ for S, we have $x = as$ and $y = bs$, and hence

   $$abs - abs\sin\theta = ab\cos\theta,$$

   which gives

   $$s = \frac{\cos\theta}{1-\sin\theta},$$

   and hence

   $$S\left(a\frac{\cos\theta}{1-\sin\theta}, b\frac{\cos\theta}{1-\sin\theta}\right).$$

   Let $\frac{x}{a} = -\frac{y}{b} = t$ for T, we have $x = at$ and $y = -bt$, and hence

   $$abt + abt\sin\theta = ab\cos\theta,$$

   which gives

   $$t = \frac{\cos\theta}{1+\sin\theta},$$

   and hence

   $$T\left(a\frac{\cos\theta}{1+\sin\theta}, -b\frac{\cos\theta}{1+\sin\theta}\right).$$

We have

$$\begin{aligned}\frac{a\frac{\cos\theta}{1-\sin\theta}+a\frac{\cos\theta}{1+\sin\theta}}{2}&=\frac{a\cos\theta}{2}\left(\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}\right)\\&=\frac{a\cos\theta}{2}\left(\frac{2}{\cos^2\theta}\right)\\&=\frac{a}{\cos\theta}\\&=a\sec\theta,\end{aligned}$$

and

$$\begin{aligned}\frac{a\frac{\cos\theta}{1-\sin\theta}-b\frac{\cos\theta}{1+\sin\theta}}{2}&=\frac{b\cos\theta}{2}\left(\frac{1}{1-\sin\theta}-\frac{1}{1+\sin\theta}\right)\\&=\frac{b\cos\theta}{2}\left(\frac{2\sin\theta}{\cos^2\theta}\right)\\&=\frac{b\sin\theta}{\cos\theta}\\&=b\tan\theta.\end{aligned}$$

This means the midpoint of ST is $(a\sec\theta, b\tan\theta)$, which is exactly P.

2. Since the tangents are perpendicular, that means

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_\theta\cdot\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_\varphi=-1,$$

and hence

$$\frac{b}{a\sin\theta}\cdot\frac{b}{a\sin\varphi}=-1,$$

which means

$$b^2=-a^2\sin\theta\sin\varphi.$$

The two tangents are

$$bx-ay\sin\theta=ab\cos\theta$$

and

$$bx-ay\sin\varphi=ab\cos\varphi.$$

Since $bx=bx$, we have

$$ay\sin\theta+ab\cos\theta=ay\sin\varphi+ab\cos\varphi,$$

and hence

$$y(\sin\theta-\sin\varphi)=b(\cos\varphi-\cos\theta),$$

which gives

$$y=b\cdot\frac{\cos\varphi-\cos\theta}{\sin\theta-\sin\varphi}.$$

Hence,

$$\begin{aligned}x&=\frac{ab\cos\theta+ay\sin\theta}{b}\\&=\frac{a}{b}\left(b\cos\theta+b\sin\theta\frac{\cos\varphi-\cos\theta}{\sin\theta-\sin\varphi}\right)\\&=a\left(\cos\theta+\sin\theta\frac{\cos\varphi-\cos\theta}{\sin\theta-\sin\varphi}\right)\\&=a\frac{\cos\theta(\sin\theta-\sin\varphi)+\sin\theta(\cos\varphi-\cos\theta)}{\sin\theta-\sin\varphi}\\&=a\cdot\frac{\sin\theta\cos\varphi-\cos\theta\sin\varphi}{\sin\theta-\sin\varphi}\\&=a\cdot\frac{\sin(\theta-\varphi)}{\sin\theta-\sin\varphi}.\end{aligned}$$

This means

$$\begin{cases} x^2 = a^2 \cdot \dfrac{\sin^2(\theta - \varphi)}{(\sin\theta - \sin\varphi)^2}, \\ y^2 = b^2 \cdot \dfrac{(\cos\varphi - \cos\theta)^2}{(\sin\theta - \sin\varphi)^2} = -a^2 \sin\theta \sin\varphi \cdot \dfrac{(\cos\varphi - \cos\theta)^2}{(\sin\theta - \sin\varphi)^2}. \end{cases}$$

Notice that

$$a^2 - b^2 = a^2 + a^2 \sin\theta \sin\varphi = a^2(1 + \sin\theta \sin\varphi).$$

Hence,

$$\begin{aligned} x^2 + y^2 &= a^2 \left[\frac{\sin^2(\theta - \varphi)}{(\sin\theta - \sin\varphi)^2} - \sin\theta \sin\varphi \cdot \frac{(\cos\varphi - \cos\theta)^2}{(\sin\theta - \sin\varphi)^2} \right] \\ &= \frac{a^2}{(\sin\theta - \sin\varphi)^2} \left[\sin^2(\theta - \varphi) - \sin\theta \sin\varphi \left(\cos\varphi - \cos\theta\right)^2 \right]. \end{aligned}$$

What is desired is to show

$$(1 + \sin\theta \sin\varphi)(\sin\theta - \sin\varphi)^2 = \sin^2(\theta - \varphi) - \sin\theta \sin\varphi \left(\cos\varphi - \cos\theta\right)^2.$$

We have

$$\begin{aligned} \text{RHS} &= (\sin\theta \cos\varphi - \cos\theta \sin\varphi)^2 - \sin\theta \sin\varphi (\cos^2\varphi + \cos^2\theta - 2\cos\varphi \cos\theta) \\ &= \sin^2\theta \cos^2\varphi + \cos^2\theta \sin^2\varphi - 2\sin\theta \cos\theta \sin\varphi \cos\varphi \\ &\quad - \sin\theta \sin\varphi \cos^2\varphi - \sin\theta \sin\varphi \cos^2\theta + 2\sin\theta \cos\theta \sin\varphi \cos\varphi \\ &= \sin\theta \cos^2\varphi (\sin\theta - \sin\varphi) + \cos^2\theta \sin\varphi (\sin\varphi - \sin\theta) \\ &= (\sin\theta \cos^2\varphi - \cos^2\theta \sin\varphi)(\sin\theta - \sin\varphi). \end{aligned}$$

Therefore, what is left to prove is that

$$(1 + \sin\theta \sin\varphi)(\sin\theta - \sin\varphi) = \sin\theta \cos^2\varphi - \cos^2\theta \sin\varphi$$

Notice that

$$\begin{aligned} \text{LHS} &= \sin\theta - \sin\varphi + \sin^2\theta \sin\varphi - \sin\theta \sin^2\varphi \\ &= \sin\theta (1 - \sin^2\varphi) - \sin\varphi (1 - \sin^2\theta) \\ &= \sin\theta \cos^2\varphi - \sin\varphi \cos^2\theta \\ &= \text{RHS}. \end{aligned}$$

This shows that

$$\frac{1}{(\sin\theta - \sin\varphi)^2} \left[\sin^2(\theta - \varphi) - \sin\theta \sin\varphi \left(\cos\varphi - \cos\theta\right)^2 \right] = 1 + \sin\theta \sin\varphi,$$

and hence

$$x^2 + y^2 = a^2 - b^2,$$

as desired.

2018.3 Question 5

1. First, we notice that
$$G_{k+1}^{k+1}=\prod_{t=1}^{k+1}a_t=a_{k+1}G_k^k,$$
and hence
$$G_{k+1}=\left(a_{k+1}G_k^k\right)^{\frac{1}{k+1}}.$$
Similarly, notice that
$$(k+1)A_{k+1}=\sum_{t=1}^{k+1}a_t=a_{k+1}+kA_k.$$
Hence,
$$\begin{aligned}(k+1)\left(A_{k+1}-G_{k+1}\right)&\geq k\left(A_k-G_k\right),\\ a_{k+1}+kA_k-(k+1)\left(a_{k+1}G_k^k\right)^{\frac{1}{k+1}}&\geq ka_k-kG_k,\\ a_{k+1}+kG_k&\geq (k+1)a_{k+1}^{\frac{1}{k+1}}G_k^{\frac{k}{k+1}}.\end{aligned}$$
Dividing both sides by G_k, we have
$$\begin{aligned}\frac{a_{k+1}}{G_k}+k&\geq (k+1)a_{k+1}^{\frac{1}{k+1}}G_k^{-\frac{1}{k+1}},\\ \lambda_k^{k+1}+k&\geq (k+1)\left(\frac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}},\\ \lambda_k^{k+1}+k&\geq (k+1)\lambda_k,\\ \lambda_k^{k+1}-(k+1)\lambda_k+k&\geq 0,\end{aligned}$$
as desired. (Notice that the condition for the equal sign is equivalent as well.)

2. By differentiation, we have
$$f'(x)=(k+1)x^k-(k+1)=(k+1)(x^k-1).$$
When $x\in(0,1), x^k\in(0,1), f'(x)<0$, and hence f is strictly decreasing.
When $x\in(1,\infty), x^k\in(1,\infty), f'(x)>0$, and hence f is strictly increasing.
Hence, $f(1)$ is the minimum for f on $(0,\infty)$. This means for all $x\in(0,\infty)$, we have
$$f(x)\geq f(1)=1^{k+1}-(k+1)+k=0,$$
taking the equal sign if and only if $x=1$.

3. (a) We show this by induction. For the base case $n=1$, $A_1=G_1=a_1$, so naturally $A_n\geq G_n$ is satisfied.
Assume that the statement holds for some $n=k$, i.e. $A_k\geq G_k$, $A_k-G_k\geq 0$. Since $k>0$ as well, we must have
$$(k+1)(A_{k+1}-G_{k+1})\geq k(A_k-G_k)\geq 0.$$
We also have $k+1>0$, and hence
$$A_{k+1}-G_{k+1}\geq 0\iff A_{k+1}\geq G_{k+1},$$
meaning the statement holds for $n=k+1$ as well.
Hence, by the principle of mathematical induction, we must have $A_n\geq G_n$ for all $n\in\mathbb{N}$, which finishes our proof.

(b) We show this by induction. For the base case $n = 1$, this condition is naturally satisfied.

Assume that the statement holds for some $n = k$, i.e. $A_k = G_k \implies a_1 = a_2 = \cdots = a_k$. We show this for $n = k + 1$. If $A_{k+1} = G_{k+1}$, then we must have

$$k(A_k - G_k) \leq (k+1)(A_{k+1} - G_{k+1}) = 0,$$

but since $A_k \geq G_k$, we must have then $A_k = G_k$, and hence the equal sign in the inequality being taken.

This must mean that

$$\lambda_k = \left(\frac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}} = 1,$$

and hence

$$a_{k+1} = G_k.$$

At the same time, since $A_k = G_k$, we must have $a_1 = a_2 = \cdots = a_k$, and hence $G_k = a_1 = a_2 = \cdots = a_k$. Therefore, we must also have

$$a_1 = a_2 = \cdots = a_k = a_{k+1},$$

which proves the statement that $A_{k+1} = G_{k+1}$ implies $a_1 = a_2 = \cdots = a_k = a_{k+1}$, which is the original statement for $n = k + 1$.

Hence, by the principle of mathematical induction, we must have $A_n = G_n$ implies $a_1 = a_2 = \cdots = a_n$ for all $n \in \mathbb{N}$, which finishes our proof.

2018.3 Question 6

1. Since A, Q, C lie on a straight line, $\mathbf{AQ} = \lambda \mathbf{AC}$ for some $\lambda \in \mathbb{R}$. This means
$$q - a = \lambda(c - a),$$
and hence
$$\frac{q-a}{c-a} = \lambda \in \mathbb{R},$$
as required.

   Hence, we must have
$$\frac{q-a}{c-a} = \left(\frac{q-a}{c-a}\right)^* = \frac{q^*-a^*}{c^*-a^*}.$$
   Cross-multiplying the terms out give
$$(c-a)(q^*-a^*) = (c^*-a^*)(q-a)$$
   exactly as desired.

   Substituting in $a^* = 1/a$ and $c^* = 1/c$, we have
$$(c-a)\left(q^* - \frac{1}{a}\right) = \left(\frac{1}{c} - \frac{1}{a}\right)(q-a),$$
   and expanding the brackets gives
$$cq^* - aq^* - \frac{c}{a} + 1 = \frac{q}{c} - \frac{a}{c} - \frac{q}{a} + 1,$$
   and hence
$$cq^* - aq^* - \frac{c}{a} = \frac{q}{c} - \frac{a}{c} - \frac{q}{a}.$$
   Multiplying by ac on both sides gives us
$$ac^2q^* - a^2cq^* - c^2 = aq - a^2 - cq,$$
   and hence
$$ac(c-a)q^* = (a-c)q - (a^2-c^2) = (a-c)q - (a-c)(a+c).$$
   We can divide through $(a-c)$ on both sides since $a \neq c$. Hence,
$$0 = q - (a+c) + acq^*,$$
   and hence
$$acq^* + q = a + c,$$
   as desired.

2. By part 1, we must have
$$acq^* + q = a + c, bdq^* + q = b + d.$$
   Since $q = q$, we have
$$acq^* - (a+c) = bdq^* - (b+d),$$
   and rearranging gives
$$(ac - bd)q^* = (a+c) - (b+d),$$
   exactly as desired.

   We also have $q^* = q^*$, and hence
$$\frac{a+c-q}{ac} = \frac{b+d-q}{bd},$$
   which gives
$$(bd)(a+c-q) = (ac)(b+d-q),$$

and rearranging gives

$$(ac - bd)q = ac(b + d) - bd(a + c).$$

Summing this with previously, we have

$$(ac - bd)(q + q^*) = (a + c) - (b + d) + ac(b + d) - bd(a + c).$$

We notice that

$$\begin{aligned}(a + c) - (b + d) + ac(b + d) - bd(a + c) &= a + c - b - d + abc + acd - abd - bcd \\ &= a - b + acd - bcd + c - d + abc - abd \\ &= (a - b)(1 + cd) + (c - d)(1 + ab),\end{aligned}$$

and hence

$$(ac - bd)(q + q^*) = (a - b)(1 + cd) + (c - d)(1 + ab),$$

exactly as desired.

3. By part 1, we must have

$$p + abp^* = a + b.$$

Since p is real, $p = p^*$, and hence

$$(1 + ab)p = a + b,$$

as desired.

Similarly, we must have

$$(1 + cd)q = c + d,$$

and putting this back into the result from part 2, we have

$$(ac - bd)(q + q^*) = \frac{(a - b)(c + d)}{p} + \frac{(c - d)(a + b)}{p},$$

and hence since $ac - bd \neq 0$, we have

$$\begin{aligned}p(q + q^*) &= \frac{(a - b)(c + d) + (c - d)(a + b)}{ac - bd} \\ &= \frac{ac + ad - bc - bd + ac + bc - ad - bd}{ac - bd} \\ &= \frac{2ac - 2bd}{ac - bd} \\ &= 2,\end{aligned}$$

as desired.

2018.3 Question 7

1. We have

$$\begin{aligned}
&\frac{(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}}{2i}\\
={}&\frac{(\cos\theta+i\sin\theta)^{2n+1}-(\cos\theta-i\sin\theta)^{2n+1}}{2i\sin^{2n+1}\theta}\\
={}&\frac{(\cos(2n+1)\theta+i\sin(2n+1)\theta)-(\cos(2n+1)\theta-i\sin(2n+1)\theta)}{2i\sin^{2n+1}\theta}\\
={}&\frac{2i\sin(2n+1)\theta}{2i\sin^{2n+1}\theta}\\
={}&\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta},
\end{aligned}$$

as desired.

By applying the binomial expansion formula on the numerator, we have

$$\begin{aligned}
&(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}\\
={}&\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot i^{2n+1-t}-\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot(-i)^{2n+1-t}\\
={}&\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot\left[i^{2n+1-t}-(-i)^{2n+1-t}\right]\\
={}&(-1)^n\cdot i\cdot\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot i^{-t}\cdot\left[1-(-1)^{1-t}\right].
\end{aligned}$$

Due to the existence of the final term, this means that only terms with even t will retain (give a 2), and odd ts will cancel. Hence,

$$\begin{aligned}
&(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}\\
={}&(-1)^n\cdot i\cdot\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot i^{-t}\cdot\left[1-(-1)^{1-t}\right]\\
={}&(-1)^n\cdot 2i\cdot\sum_{t=0}^{n}\binom{2n+1}{2t}\cot^{2t}\theta\cdot i^{-2t}\\
={}&2i(-1)^n\cdot\sum_{t=0}^{n}\binom{2n+1}{2t}\cot^{2t}\theta\cdot(-1)^t\\
={}&2i(-1)^n\cdot\sum_{t=0}^{n}\binom{2n+1}{2n-2t+1}\cot^{2t}\theta\cdot(-1)^t\\
={}&2i(-1)^n\cdot\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^{n-t}\\
={}&2i\cdot\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^t.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}\\
={}&\frac{2i\cdot\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^t}{2i}\\
={}&\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^t.
\end{aligned}$$

The left-hand side of the original equation is

$$\sum_{t=0}^{n} \binom{2n+1}{2t+1} x^{n-t} \cdot (-1)^t.$$

Let $x = \cot^2 \theta$, we have

$$\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta} = \sum_{t=0}^{n} \binom{2n+1}{2t+1} x^{n-t} \cdot (-1)^t = 0.$$

Therefore, we have $\sin(2n+1)\theta = 0$, and hence $(2n+1)\theta = m\pi$ for $m \in \mathbb{Z}$.

To avoid duplicate solutions for $x = \cot^2 \theta$, we restrict $\theta \in \left(0, \frac{\pi}{2}\right]$, and hence $(2n+1)\theta \in \left(0, \left(n+\frac{1}{2}\right)\pi\right]$, and hence $m = 1, 2, \ldots, n$.

This solves to $\theta = \frac{m\pi}{2n+1}$ for $m = 1, 2, \ldots, n$, and hence this gives exactly

$$x = \cot^2\left(\frac{m\pi}{2n+1}\right).$$

2. By Vieta's Theorem, we will have

$$\sum_{m=1}^{n} x_m = -\frac{-\binom{2n+1}{3}}{\binom{2n+1}{1}} = \frac{(2n+1)(2n)(2n-1)}{(2n+1)\cdot 3 \cdot 2 \cdot 1} = \frac{n(2n-1)}{3},$$

and since we have

$$x_m = \cot^2\left(\frac{m\pi}{2n+1}\right),$$

we have

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{n(2n-1)}{3}.$$

3. For $0 < \theta < \frac{1}{2}\pi$, we have $0 < \sin\theta < \theta < \tan\theta$, and squaring this gives

$$0 < \sin^2\theta < \theta^2 < \tan^2\theta,$$

and flipping to the reciprocal gives

$$0 < \cot^2\theta < \frac{1}{\theta^2} < \csc^2\theta = 1 + \cot^2\theta,$$

which proves exactly what is desired.

Therefore, we have

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) < \sum_{m=1}^{n} \frac{1}{\left(\frac{m\pi}{2n+1}\right)^2} < \sum_{m=1}^{n} \left[1 + \cot^2\left(\frac{m\pi}{2n+1}\right)\right],$$

and hence

$$\frac{n(2n-1)}{3} < \sum_{m=1}^{n} \frac{(2n+1)^2}{m^2\pi^2} < \frac{2n(n+1)}{3},$$

and hence

$$\frac{n(2n-1)\pi^2}{3(2n+1)^2} < \sum_{m=1}^{n} \frac{1}{m^2} < \frac{2n(n+1)\pi^2}{3(2n+1)^2}.$$

Take the limit as $n \to \infty$, the strict inequalities become weak, and hence

$$\lim_{n\to\infty} \frac{n(2n-1)\pi^2}{3(2n+1)^2} \leq \sum_{m=1}^{\infty} \frac{1}{m^2} \leq \lim_{n\to\infty} \frac{2n(n+1)\pi^2}{3(2n+1)^2},$$

and hence

$$\frac{2\pi^2}{3 \cdot 2^2} \leq \sum_{m=1}^{\infty} \frac{1}{m^2} \leq \frac{2n\pi^2}{3 \cdot 2^2},$$

and therefore

$$\frac{\pi^2}{6} \leq \sum_{m=1}^{\infty} \frac{1}{m^2} \leq \frac{\pi^2}{6},$$

and hence

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6},$$

as desired.

2018.3 Question 8

1. Using the substitution $t = \frac{1}{x}$, we have

$$\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{1}{x^2} \implies \mathrm{d}x = -x^2 \, \mathrm{d}t = -\frac{\mathrm{d}t}{t^2},$$

and when $x \to 0^+$, $t \to \infty$, and when $x = 1$, $t = 1$. Hence,

$$\begin{aligned} I &= \int_0^1 \frac{f(x^{-1})}{1+x} \, \mathrm{d}x \\ &= \int_1^\infty \frac{f(t)}{1+t^{-1}} \cdot \left(-\frac{\mathrm{d}t}{t^2}\right) \\ &= \int_1^\infty \frac{f(t) \, \mathrm{d}t}{t(1+t)} \\ &= \int_1^2 \frac{f(t) \, \mathrm{d}t}{t(1+t)} + \int_2^3 \frac{f(t) \, \mathrm{d}t}{t(1+t)} + \int_3^4 \frac{f(t) \, \mathrm{d}t}{t(1+t)} + \cdots \\ &= \sum_{n=1}^\infty \int_n^{n+1} \frac{f(t) \, \mathrm{d}t}{t(1+t)}, \end{aligned}$$

as desired.

Since $f(x) = f(x+1)$ for all x, we must have that $f(x) = f(x+n)$ for all x and integers n. Also, we have

$$\frac{1}{y(1+y)} = \frac{1}{y} - \frac{1}{1+y}.$$

Hence,

$$\begin{aligned} I &= \sum_{n=1}^\infty \int_n^{n+1} \frac{f(t) \, \mathrm{d}t}{t(1+t)} \\ &= \sum_{n=1}^\infty \int_0^1 \frac{f(n+t) \, \mathrm{d}t}{(n+t)(n+t+1)} \\ &= \sum_{n=1}^\infty \int_0^1 f(t) \cdot \left[\frac{1}{n+t} - \frac{1}{n+t+1}\right] \mathrm{d}t \\ &= \sum_{n=1}^\infty \int_0^1 \frac{f(t) \, \mathrm{d}t}{n+t} - \sum_{n=1}^\infty \int_0^1 \frac{f(t) \, \mathrm{d}t}{n+t+1} \\ &= \sum_{n=1}^\infty \int_0^1 \frac{f(t) \, \mathrm{d}t}{n+t} - \sum_{n=2}^\infty \int_0^1 \frac{f(t) \, \mathrm{d}t}{n+t} \\ &= \int_0^1 \frac{f(t) \, \mathrm{d}t}{1+t}. \end{aligned}$$

2. For the first integral, simply consider $f(x) = \{x\}$, and we can immediately see that $f(x)$ has period of 1 from the definition. Hence,

$$\int_0^1 \frac{\{x^{-1}\}}{1+x} \, \mathrm{d}x = \int_0^1 \frac{f(x^{-1})}{1+x} \, \mathrm{d}x = \int_0^1 \frac{f(x)}{1+x} \, \mathrm{d}x = \int_0^1 \frac{\{x\}}{1+x} \, \mathrm{d}x.$$

Since for $0 < x < 1$, we have $\{x\} = x$, and hence

$$\begin{aligned}\int_0^1 \frac{\{x^{-1}\}}{1+x}\,\mathrm{d}x &= \int_0^1 \frac{\{x\}}{1+x}\,\mathrm{d}x \\ &= \int_0^1 \frac{x}{1+x}\,\mathrm{d}x \\ &= \int_0^1 \left(1 - \frac{1}{1+x}\right)\mathrm{d}x \\ &= 1 - [\ln(1+x)]_0^1 \\ &= 1 - (\ln(2) - \ln(1)) \\ &= 1 - \ln 2.\end{aligned}$$

For the second integral, we let $g(x) = \{2x\}$, and we can see that $g(x)$ has a period of $\frac{1}{2}$, and hence it also has a period of 1. Hence,

$$\int_0^1 \frac{\{2x^{-1}\}}{1+x}\,\mathrm{d}x = \int_0^1 \frac{g(x^{-1})}{1+x}\,\mathrm{d}x = \int_0^1 \frac{g(x)}{1+x}\,\mathrm{d}x = \int_0^1 \frac{\{2x\}}{1+x}\,\mathrm{d}x.$$

We split this integral into two parts, $[0, 0.5]$ and $[0.5, 1]$.

$$\begin{aligned}\int_0^1 \frac{\{2x^{-1}\}}{1+x}\,\mathrm{d}x &= \int_0^1 \frac{\{2x\}}{1+x}\,\mathrm{d}x \\ &= \int_0^{0.5} \frac{\{2x\}}{1+x}\,\mathrm{d}x + \int_{0.5}^1 \frac{\{2x\}}{1+x}\,\mathrm{d}x \\ &= \int_0^{0.5} \frac{2x}{1+x}\,\mathrm{d}x + \int_{0.5}^1 \frac{2x-1}{1+x}\,\mathrm{d}x \\ &= \int_0^{0.5} \left[2 - \frac{2}{1+x}\right]\mathrm{d}x + \int_{0.5}^1 \left[2 - \frac{3}{1+x}\right]\mathrm{d}x \\ &= 1 - 2\left[\ln(1+x)\right]_0^{0.5} + 1 - 3\left[\ln(1+x)\right]_{0.5}^1 \\ &= 2 - 2\ln 1.5 + 2\ln 1 - 3\ln 2 + 3\ln 1.5 \\ &= 2 - 3\ln 2 + \ln 1.5 \\ &= 2 - 3\ln 2 + \ln 3 - \ln 2 \\ &= 2 - 4\ln 2 + \ln 3.\end{aligned}$$

2018.3 Question 12

1. $\mathrm{P}(Y_k) \leq y$ is the probability that there is at least k numbers that are less than equal to y.

   If there are $k \leq m \leq n$ numbers less than or equal to y, then there must be $n - m$ numbers greater than or equal to y. The probability of the first thing happening for each number is y, and for the second thing happening for each number is $1 - y$. We also have to choose m numbers from the n to make them less than or equal to y. Therefore,

$$\mathrm{P}(Y_k \leq y) = \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}.$$

2. We have

$$m\binom{n}{m} = m \cdot \frac{n!}{m!(n-m)!} = \frac{n!}{(m-1)!(n-m)!} = n \cdot \frac{(n-1)!}{(m-1)!(n-m)!} = n\binom{n-1}{m-1}.$$

   We have

$$(n-m)\binom{n}{m} = (n-m) \cdot \frac{n!}{m!(n-m)!} = \frac{n!}{m!(n-m-1)!} = n \cdot \frac{(n-1)!}{m!(n-m-1)!} = n\binom{n-1}{m}.$$

   The cumulative distribution function F_{Y_k} is

$$F_{Y_k}(y) = \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}.$$

   Therefore, the probability density function f_{Y_k} is

$$\begin{aligned}
f_{Y_k}(y) &= F'_{Y_k}(y) \\
&= \sum_{m=k}^{n} \binom{n}{m} \left[m y^{m-1} (1-y)^{n-m} - (n-m) y^m (1-y)^{n-m-1} \right] \\
&= \sum_{m=k}^{n} y^{m-1} (1-y)^{n-m-1} \left[m\binom{n}{m}(1-y) - (n-m)\binom{n}{m} y \right] \\
&= n \left[\sum_{m=k}^{n} \binom{n-1}{m-1} y^{m-1} (1-y)^{n-m} - \sum_{m=k}^{n-1} \binom{n-1}{m} y^m (1-y)^{n-m-1} \right] \\
&= n \left[\sum_{m=k}^{n} \binom{n-1}{m-1} y^{m-1} (1-y)^{n-m} - \sum_{m=k+1}^{n} \binom{n-1}{m-1} y^{m-1} (1-y)^{n-m} \right] \\
&= n\binom{n-1}{k-1} y^{k-1} (1-y)^{n-k}.
\end{aligned}$$

   Since $Y_k \in [0, 1]$, we must have

$$\int_0^1 f_{Y_k}(y) \, \mathrm{d}y = 1,$$

   and hence

$$n\binom{n-1}{k-1} \int_0^1 y^{k-1} (1-y)^{n-k} \, \mathrm{d}y = 1,$$

   and therefore we have

$$\int_0^1 y^{k-1} (1-y)^{n-k} \, \mathrm{d}y = \frac{1}{n\binom{n-1}{k-1}}.$$

3. By the definition of the expectation,

$$\begin{aligned}
\mathrm{E}(Y_k) &= \int_0^1 y f_{Y_k}(y)\,\mathrm{d}y \\
&= n\binom{n-1}{k-1}\int_0^1 y^k(1-y)^{n-k}\,\mathrm{d}y \\
&= n\binom{n-1}{k-1}\cdot\frac{1}{(n+1)\binom{n}{k}} \\
&= \frac{n\cdot\frac{(n-1)!}{(k-1)!(n-k)!}}{(n+1)\cdot\frac{n!}{k!(n-k)!}} \\
&= \frac{\frac{n!}{(k-1)!(n-k)!}}{\frac{(n+1)n!}{k(k-1)!(n-k)!}} \\
&= \frac{k}{n+1}.
\end{aligned}$$

2018.3 Question 13

By the definition of a probability generating function, we have

$$G(1) = \sum_{n=0}^{\infty} \mathrm{P}(X = n), \text{ and } G(-1) = \sum_{n=0}^{\infty} (-1)^n \mathrm{P}(X = n).$$

Hence,

$$G(1) + G(-1) = \sum_{n=0}^{\infty} [1 + (-1)^n] \mathrm{P}(X = n).$$

When n is odd, $1 + (-1)^n = 0$. When n is even, $1 + (-1)^n = 2$.
This means

$$G(1) + G(-1) = 2 \sum_{n=0}^{\infty} \mathrm{P}(X = 2n),$$

which gives

$$\frac{1}{2}(G(1) + G(-1)) = \sum_{n=0}^{\infty} \mathrm{P}(X = 2n) = \mathrm{P}(X = 0 \text{ or } X = 2 \text{ or } X = 4 \ldots).$$

Since $X \sim \mathrm{Po}(\lambda)$, we have

$$\mathrm{P}(X = x) = e^{-\lambda} \frac{\lambda^x}{x!},$$

and hence the probability generating function for X, $G(t)$, must satisfy

$$\begin{aligned} G(t) &= \sum_{n=0}^{\infty} \mathrm{P}(X = n) \cdot t^n \\ &= \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \cdot t^n \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \\ &= e^{-\lambda} \cdot e^{\lambda t} \\ &= e^{-\lambda(1-t)}. \end{aligned}$$

1. Consider $G(t) + G(-t)$. By definition, we have

$$G(t) = \sum_{n=0}^{\infty} \mathrm{P}(X = n) t^n, G(-t) = \sum_{n=0}^{\infty} (-1)^n \mathrm{P}(X = n) t^n,$$

and hence

$$G(t) + G(-t) = \sum_{n=0}^{\infty} (1 + (-1)^n) \mathrm{P}(X = n) t^n = 2 \sum_{n=0}^{\infty} \mathrm{P}(X = 2n) t^{2n}.$$

Let $H(t)$ be the probability generating function of Y, we have

$$\begin{aligned} H(t) &= \sum_{n=0}^{\infty} \mathrm{P}(Y = n) \cdot t^n \\ &= \sum_{n=0}^{\infty} \mathrm{P}(Y = 2n) \cdot t^{2n} \\ &= \sum_{n=0}^{\infty} k \mathrm{P}(X = 2n) \cdot t^{2n} \\ &= \frac{k}{2} (G(t) + G(-t)). \end{aligned}$$

To find k, we must have $H(1) = 1$. Hence,

$$1 = \frac{k}{2}\left(G(1) + G(-1)\right) = \frac{k}{2}\left(e^{-\lambda(1-1)} + e^{-\lambda(1+1)}\right) = \frac{k}{2}\left(1 + e^{-2\lambda}\right),$$

which gives

$$k = \frac{2}{1 + e^{-2\lambda}} = \frac{2e^{\lambda}}{e^{\lambda} + e^{-\lambda}} = \frac{e^{\lambda}}{\cosh\lambda}.$$

Hence,

$$\begin{aligned} H(t) &= \frac{k}{2}\left(G(t) + G(-t)\right) \\ &= \frac{e^{\lambda}}{2\cosh\lambda}\left(e^{-\lambda(1-t)} + e^{-\lambda(1+t)}\right) \\ &= \frac{1}{\cosh\lambda}\frac{e^{\lambda t} + e^{-\lambda t}}{2} \\ &= \frac{\cosh\lambda t}{\cosh\lambda}. \end{aligned}$$

Differentiating this with respect to t, we have

$$H'(t) = \frac{\lambda\sinh\lambda t}{\cosh\lambda},$$

and hence

$$\mathrm{E}(Y) = H'(1) = \frac{\lambda\sinh\lambda\cdot 1}{\cosh\lambda} = \lambda\tanh\lambda.$$

Since $-1 < \tanh\lambda < 1$, we have $\lambda\tanh\lambda < \lambda$, and so $\mathrm{E}(Y) < \lambda$ for $\lambda > 0$.

2. Consider $G(t) + G(-t) + G(it) + G(-it)$. By definition, we have

$$G(t) + G(-t) + G(it) + G(-it) = \sum_{n=0}^{\infty}\left(1 + (-1)^n + i^n + (-i)^n\right)\mathrm{P}(X = n)\cdot t^n.$$

Let m be an integer. Consider the following four cases:

- $n = 4m$, $1 + (-1)^n + i^n + (-i)^n = 1 + 1 + 1 + 1 = 4$.
- $n = 4m + 1$, $1 + (-1)^n + i^n + (-i)^n = 1 + (-1) + i + (-i) = 0$.
- $n = 4m + 2$, $1 + (-1)^n + i^n + (-i)^n = 1 + 1 + (-1) + (-1) = 0$.
- $n = 4m + 3$, $1 + (-1)^n + i^n + (-i)^n + 1 + (-1) + (-i) + i = 0$.

Hence,

$$G(t) + G(-t) + G(it) + G(-it) = 4\sum_{n=0}^{\infty}\mathrm{P}(X = 4n)\cdot t^{4n}.$$

Let $P(t)$ be the probability generating function of Z, we have

$$\begin{aligned} P(t) &= \sum_{n=0}^{\infty}\mathrm{P}(Z = n)\cdot t^n \\ &= \sum_{n=0}^{\infty}\mathrm{P}(Z = 4n)\cdot t^{4n} \\ &= c\sum_{n=0}^{\infty}\mathrm{P}(X = 4n)\cdot t^{4n} \\ &= \frac{c}{4}\left(G(t) + G(-t) + G(it) + G(-it)\right). \end{aligned}$$

Since $P(1) = 0$, we must have

$$\begin{aligned}
1 &= \frac{c}{4}\left(G(1) + G(-1) + G(i) + G(-i)\right) \\
&= \frac{c}{4}\left(e^{-\lambda(1-1)} + e^{-\lambda(1+1)} + e^{-\lambda(1-i)} + e^{-\lambda(1+i)}\right) \\
&= \frac{ce^{-\lambda}}{4}\left(e^{\lambda} + e^{-\lambda} + e^{i\lambda} + e^{-i\lambda}\right) \\
&= \frac{ce^{-\lambda}}{2}\left(\cos\lambda + \cosh\lambda\right).
\end{aligned}$$

Hence,

$$c = \frac{2e^{\lambda}}{\cos\lambda + \cosh\lambda}.$$

Therefore,

$$\begin{aligned}
P(t) &= \frac{c}{4}\left(G(t) + G(-t) + G(it) + G(-it)\right) \\
&= \frac{e^{\lambda}}{2(\cos\lambda + \cosh\lambda)}\left[e^{-\lambda(1-t)} + e^{-\lambda(1+t)} + e^{-\lambda(1-it)} + e^{-\lambda(1+it)}\right] \\
&= \frac{e^{\lambda t} + e^{-\lambda t} + e^{\lambda it} + e^{-\lambda it}}{2(\cos\lambda + \cosh\lambda)} \\
&= \frac{\cos\lambda t + \cosh\lambda t}{\cos\lambda + \cosh\lambda}.
\end{aligned}$$

Differentiating this with respect to t gives us

$$P'(t) = \frac{\lambda(-\sin\lambda t + \sinh\lambda t)}{\cos\lambda + \cosh\lambda},$$

and hence

$$\mathrm{E}(Z) = P'(1) = \frac{\lambda(-\sin\lambda + \sinh\lambda)}{\cos\lambda + \cosh\lambda}.$$

$\mathrm{E}(Z) < \lambda$ is equivalent to

$$\frac{\sinh\lambda - \sin\lambda}{\cosh\lambda + \cos\lambda} < 1,$$

which is then equivalent to

$$\sinh\lambda - \cosh\lambda < \sin\lambda + \cos\lambda,$$

which is

$$-e^{-\lambda} < \sin\lambda + \cos\lambda.$$

However, this is not necessarily true. Let $\lambda = \pi$. We have

$$\text{LHS} = -e^{-\pi} > -e^{0} = -1,$$

and

$$\text{RHS} = \sin\pi + \cos\pi = -1,$$

which means LHS > RHS for $\lambda = \pi$, which means $\mathrm{E}(Z) > \lambda$. Therefore, the statement is not true.