# 2017 Paper 2

## 2017.2 Question 1

1. Using integration by parts, we notice that

$$\begin{aligned}
(n+1)I_n &= (n+1)\int_0^1 x^n \arctan x \,\mathrm{d}x \\
&= \int_0^1 \arctan x \,\mathrm{d}x^{n+1} \\
&= \left[\arctan x \cdot x^{n+1}\right]_0^1 - \int_0^1 x^{n+1} \,\mathrm{d}\arctan x \\
&= \arctan 1 \cdot 1^{n+1} - \arctan 0 \cdot 0^{n+1} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,\mathrm{d}x \\
&= \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,\mathrm{d}x.
\end{aligned}$$

Set $n = 0$, and we have

$$\begin{aligned}
I_0 &= (0+1)I_0 \\
&= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x \\
&= \frac{\pi}{4} - \frac{1}{2} \cdot \left[\ln(1+x^2)\right]_0^1 \\
&= \frac{\pi}{4} - \frac{1}{2} \cdot [\ln 2 - \ln 1] \\
&= \frac{\pi}{4} - \frac{\ln 2}{2}.
\end{aligned}$$

2. Using the result in the previous part,

$$\begin{aligned}
(n+3)I_{n+2} + (n+1)I_n &= \left(\frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}\,\mathrm{d}x\right) + \left(\frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,\mathrm{d}x\right) \\
&= \frac{\pi}{2} - \int_0^1 \frac{x^{n+1}+x^{n+3}}{1+x^2}\,\mathrm{d}x \\
&= \frac{\pi}{2} - \int_0^1 \frac{x^{n+1}\left(1+x^2\right)}{1+x^2}\,\mathrm{d}x \\
&= \frac{\pi}{2} - \int_0^1 x^{n+1}\,\mathrm{d}x \\
&= \frac{\pi}{2} - \frac{1}{n+2}\left[x^{n+2}\right]_0^1 \\
&= \frac{\pi}{2} - \frac{1}{n+2}.
\end{aligned}$$

Letting $n = 0$, and we have

$$3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2}.$$

Letting $n = 2$, and we have

$$5I_4 + 3I_2 = \frac{\pi}{2} - \frac{1}{4}.$$

Subtracting the first one from the second one, and hence

$$5I_4 - I_0 = \frac{1}{4}.$$

Hence,

$$I_4 = \frac{1}{5} \cdot \left[\frac{1}{4} + \left(\frac{\pi}{4} - \frac{\ln 2}{2}\right)\right] = \frac{1}{20} + \frac{\pi}{20} - \frac{\ln 2}{10}.$$

3. Let $n = 1$, and the statement says

$$\begin{aligned}(4n+1)I_{4n} &= 5I_4 \\ &= A - \frac{1}{2}\sum_{r=1}^{2\cdot 1}(-1)^r\frac{1}{r} \\ &= A - \frac{1}{2}\left(-\frac{1}{1}+\frac{1}{2}\right) \\ &= A + \frac{1}{4}.\end{aligned}$$

Comparing to the previous expression, we claim that

$$A = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

This shows the base case for $n = 1$. For the induction step, we first introduce a lemma. Since

$$(n+5)I_{n+4} + (n+3)I_{n+2} = \frac{\pi}{2} - \frac{1}{n+4}, (n+3)I_{n+2} + (n+1)I_n = \frac{\pi}{2} - \frac{1}{n+2},$$

subtracting the second one from the first one will give us

$$(n+5)I_{n+4} - (n+1)I_n = \frac{1}{n+2} - \frac{1}{n+4}.$$

Setting $n = 4m$, we have

$$\begin{aligned}(4(m+1)+1)I_{4(m+1)} &= (4m+1)I_{4m} + \frac{1}{4m+2} - \frac{1}{4m+4} \\ &= (4m+1)I_{4m} - \frac{1}{2}\cdot\left(-\frac{1}{2m+1}+\frac{1}{2m+2}\right) \\ &= (4m+1)I_{4m} - \frac{1}{2}\cdot\left[(-1)^{2m+1}\frac{1}{2m+1} + (-1)^{2m+2}\frac{1}{2m+2}\right].\end{aligned}$$

Now we show the inductive step. Assume the statement is true for some $n = k \geq 1$, i.e.

$$(4k+1)I_{4k} = A - \frac{1}{2}\sum_{r=1}^{2n}(-1)^r\frac{1}{r}.$$

Using the identity above, we have

$$\begin{aligned}(4(k+1)+1)I_{4(k+1)} &= (4k+1)I_{4k} - \frac{1}{2}\cdot\left[(-1)^{2k+1}\frac{1}{2k+1} + (-1)^{2k+2}\frac{1}{2k+2}\right] \\ &= A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r\frac{1}{r} - \frac{1}{2}\cdot\left[(-1)^{2k+1}\frac{1}{2k+1} + (-1)^{2k+2}\frac{1}{2k+2}\right] \\ &= A - \frac{1}{2}\sum_{r=1}^{2(k+1)}(-1)^r\frac{1}{r}.\end{aligned}$$

Hence, the original statement is true for $n = 1$ (as shown when determining the value of $A$), and given the original statement holds for some $n = k \geq 1$, it holds for $n = k + 1$. By the principle of mathematical induction, this statement holds for all $n \geq 1$, where

$$A = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

## 2017.2 Question 2

We have

$$\begin{aligned}
x_{n+2} &= \frac{ax_{n+1}-1}{x_{n+1}+b} \\
&= \frac{a\cdot\frac{ax_n-1}{x_n+b}-1}{\frac{ax_n-1}{x_n+b}+b} \\
&= \frac{a(ax_n-1)-(x_n+b)}{(ax_n-1)+b(x_n+b)} \\
&= \frac{(a^2-1)x_n-(a+b)}{(a+b)x_n+(b^2-1)}.
\end{aligned}$$

1. If the sequence is periodic with period 2, then for all integers $n \geq 0$, we have

$$\begin{aligned}
x_{n+2} = x_n &\iff x_n\left[(a+b)x_n+(b^2-1)\right] = (a^2-1)x_n-(a+b) \\
&\iff (a+b)x_n^2-(a+b)(a-b)x_n+(a+b) = 0 \\
&\iff (a+b)(x_n^2-(a-b)x_n+1) = 0.
\end{aligned}$$

   We also have

$$\begin{aligned}
x_{n+1} = x_n &\iff x_n(x_n+b) = ax_n-1 \\
&\iff x_n^2-(a-b)x_b+1 = 0,
\end{aligned}$$

   and this means that for some $n = k \geq 0$, we must have $x_n^2-(a-b)x_n+1 \neq 0$ (otherwise, the sequence will have period 1).

   Therefore, for such $n = k$, we must have $a+b = 0$ for the first condition to be true, and hence this is a necessary condition.

2. Using the formula between $x_{n+4}$ and $x_n$, we have

$$\begin{aligned}
x_{n+4} &= \frac{(a^2-1)x_{n+2}-(a+b)}{(a+b)x_{n+2}+(b^2-1)} \\
&= \frac{(a^2-1)\cdot\frac{(a^2-1)x_n-(a+b)}{(a+b)x_n+(b^2-1)}-(a+b)}{(a+b)\cdot\frac{(a^2-1)x_n-(a+b)}{(a+b)x_n+(b^2-1)}+(b^2-1)} \\
&= \frac{(a^2-1)\cdot\left[(a^2-1)x_n-(a+b)\right]-(a+b)\cdot\left[(a+b)x_n+(b^2-1)\right]}{(a+b)\cdot[(a^2-1)x_n-(a+b)]+(b^2-1)\cdot[(a+b)x_n+(b^2-1)]} \\
&= \frac{\left[(a^2-1)^2-(a+b)^2\right]x_n-\left[(a^2-1)(a+b)+(a+b)(b^2-1)\right]}{(a+b)\left[(a^2-1)+(b^2-1)\right]x_n+\left[(b^2-1)^2-(a+b)^2\right]}.
\end{aligned}$$

   If sequence has period 4, we have $x_{n+4} = x_n$ for all integers $n \geq 0$, and the sequence does not have period 1, 2 or 3.

   We notice

$$\begin{aligned}
x_{n+4} = x_n &\iff x_n\cdot\left[(a+b)\left[(a^2-1)+(b^2-1)\right]x_n+\left[(b^2-1)^2-(a+b)^2\right]\right] \\
&\qquad = \left[(a^2-1)^2-(a+b)^2\right]x_n-\left[(a^2-1)(a+b)+(a+b)(b^2-1)\right] \\
&\iff (a+b)(a^2+b^2-2)\left(x_n^2-(a-b)x_n+1\right) = 0.
\end{aligned}$$

   From the previous part, we know that for some $n = k \geq 0$, we must have

$$(a+b)\left(x_k^2-(a-b)x_k+1\right) \neq 0,$$

   which means $a+b \neq 0$ and $x_k^2-(a-b)x_k+1 \neq 0$. Hence, we must have $a^2+b^2-2 = 0$.

   On the other hand, if $a^2+b^2-2 = 0$, $a+b \neq 0$ and $x_k^2-(a-b)x_k+1 \neq 0$ for some $n = k \geq 0$, we know that the sequence does not satisfy $x_{n+1} = x_n$, does not satisfy $x_{n+2} = x_n$, and it satisfies $x_{n+4} = x_n$.

If $x_{n+3} = x_n$, then we have $x_{n+3} = x_{n+4}$ which contradicts with not satisfying $x_{n+1} = x_n$. Hence, the sequence does not satisfy $x_{n+3} = x_n$, and it must have period 4.

Therefore, the sequence has period 4, if and only if

$$\begin{cases} a + b \neq 0, \\ a^2 + b^2 - 2 = 0, \\ x_k^2 - (a - b)x_k + 1 \neq 0 \text{ for some } n = k \geq 0. \end{cases}$$

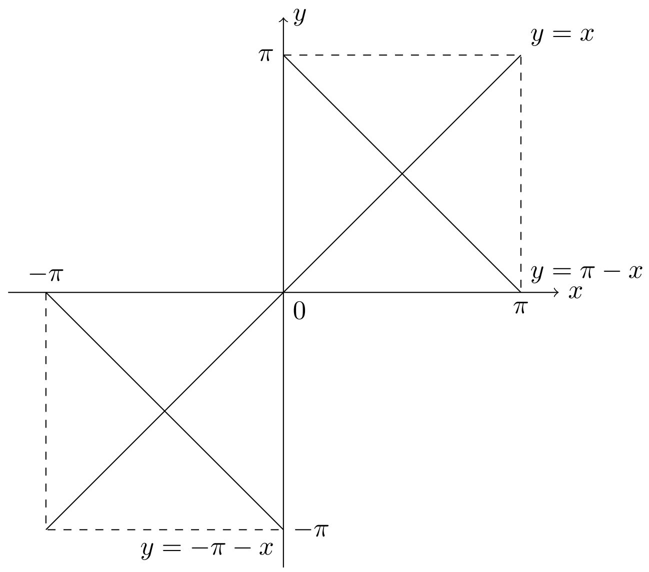
## 2017.2 Question 3

1. Since $\sin y = \sin x$, we must have
$$y = x + 2k\pi$$
where $k \in \mathbb{Z}$, or
$$y = (2k+1)\pi - x$$
where $k \in \mathbb{Z}$.

For the first case, since $x \in [-\pi, \pi]$ and $y \in [-\pi, \pi]$, we must have simply $x = y$.

For the second case, within this range, we can have $y = \pi - x$, and $y = -\pi - x$.

Hence, the sketch looks as follows.




2. Differentiating with respect to $x$, we have
$$\cos y \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\cos x.$$
Since $\sin y = \frac{1}{2}\sin x$, $\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - \frac{1}{4}\sin^2 x}$. Since $0 \le y \le \frac{1}{2}\pi$, $\cos y > 0$, and hence $\cos y = \frac{1}{2}\sqrt{4 - \sin^2 x}$. Hence,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{1}{2}\cos x}{\frac{1}{2}\sqrt{4 - \sin^2 x}} = \frac{\cos x}{\sqrt{4 - \sin^2 x}}.$$
Differentiating this again gives us
$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{(-\sin x)\sqrt{4 - \sin^2 x} - \frac{1}{2} \cdot (-2\sin x) \cdot \cos x \cdot \frac{1}{\sqrt{4-\sin^2 x}} \cdot \cos x}{4 - \sin^2 x} \\ &= \frac{(-\sin x)(4 - \sin^2 x) + \sin x \cos^2 x}{(4 - \sin^2 x)^{\frac{3}{2}}} \\ &= \frac{-4\sin x + \sin^3 x + \sin x(1 - \sin^2 x)}{(4 - \sin^2 x)^{\frac{3}{2}}} \\ &= -\frac{3\sin x}{(4 - \sin^2 x)^{\frac{3}{2}}},\end{aligned}$$
as desired.

Within this range of $x$ and $y$, we have

$$y = \arcsin\left(\frac{1}{2}\sin x\right),$$

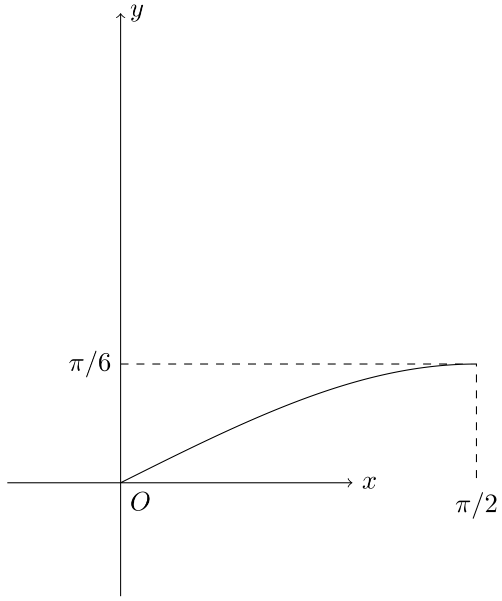
and hence this is a function, and each $x$ corresponds to a unique $y$.
At $x = 0$,

$$y = 0, y' = \frac{\cos 0}{\sqrt{4 - \sin^2 0}} = \frac{1}{2},$$

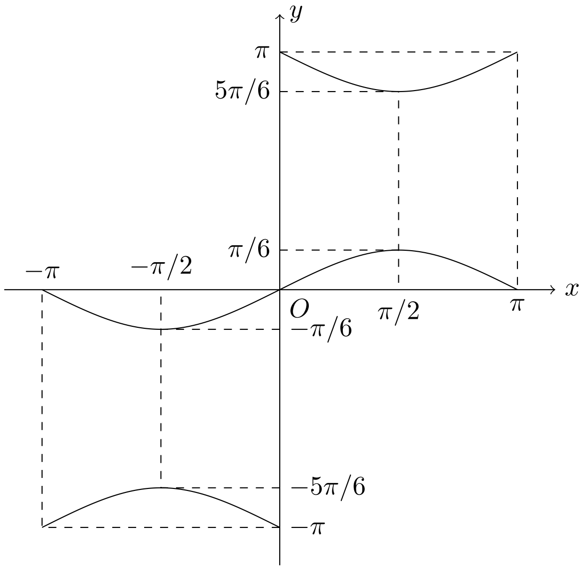
and at $x = \frac{\pi}{2}$,

$$y = \frac{\pi}{6}, y' = -\frac{\cos\frac{\pi}{2}}{\sqrt{4 - \sin^2\frac{\pi}{2}}} = 0.$$

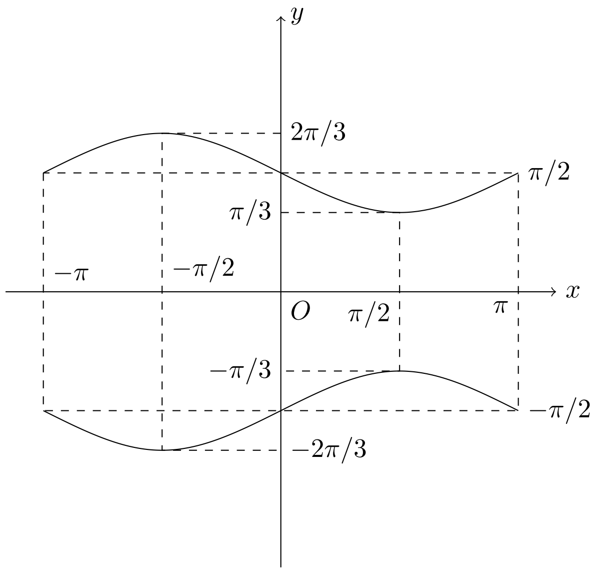
Since $y'' = -\frac{3\sin x}{(4-\sin^2 x)^{\frac{3}{2}}} < 0$ for $x \in \left[0, \frac{\pi}{2}\right]$, this function is concave, and hence the graph looks as follows.




Hence, for $(x, y) \in [-\pi, \pi]^2$, the graph looks as follows, by symmetry.




3. The graph is as follows.


y
2π/3
π/2
π/3
−π
−π/2
x
O
π/2
π
−π/3
−π/2
−2π/3

## 2017.2 Question 4

1. If $f(x) = 1$, this gives

$$\left(\int_a^b g(x)\,\mathrm{d}x\right)^2 \leq (b-a)\left(\int_a^b g(x)^2\,\mathrm{d}x\right).$$

Let $g(x) = e^x$, $a = 0$ and $b = t$, and we have

$$\text{LHS} = \left(\int_0^t e^x\,\mathrm{d}x\right)^2 = \left(e^t - 1\right)^2,$$

and

$$\text{RHS} = t\int_0^t e^{2x}\,\mathrm{d}x = \frac{t}{2}\left(e^{2t} - 1\right) = \frac{t}{2}\left(e^t - 1\right)\left(e^t + 1\right).$$

Since LHS $\leq$ RHS, we have

$$\left(e^t - 1\right)^2 \leq \frac{t}{2}\left(e^t - 1\right)\left(e^t + 1\right),$$

and hence

$$\frac{e^t - 1}{e^t + 1} \leq \frac{t}{2},$$

since $e^t + 1 > 0$.

2. If $f(x) = x$, and $a = 0, b = 1$, the Schwarz inequality gives

$$\left(\int_0^1 xg(x)\,\mathrm{d}x\right)^2 \leq \int_0^1 x^2\,\mathrm{d}x \int_0^1 g(x)^2\,\mathrm{d}x.$$

Since

$$\int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3},$$

we therefore have

$$3\left(\int_0^1 xg(x)\,\mathrm{d}x\right)^2 \leq \int_0^1 g(x)^2\,\mathrm{d}x.$$

Consider $g(x) = \exp\left(-\frac{1}{4}x^2\right)$. Notice that

$$\begin{aligned}\int_0^1 xg(x)\,\mathrm{d}x &= \int_0^1 x\exp\left(-\frac{1}{4}x^2\right)\mathrm{d}x \\ &= -2\left[\exp\left(-\frac{1}{4}x^2\right)\right]_0^1 \\ &= -2\left[\exp\left(-\frac{1}{4}\right) - \exp(0)\right] \\ &= 2\left(1 - \exp\left(-\frac{1}{4}\right)\right),\end{aligned}$$

and hence

$$3 \cdot \left[2\left(1 - \exp\left(-\frac{1}{4}\right)\right)\right]^2 \leq \int_0^1 \exp\left(-\frac{1}{2}x^2\right)\mathrm{d}x,$$

which is equivalent to

$$\int_0^1 \exp\left(-\frac{1}{2}x^2\right)\mathrm{d}x \geq 12\left(1 - \exp\left(-\frac{1}{4}\right)\right)^2,$$

as desired.

3. For the right-half of the inequality, let $f(x) = 1$, and let the bounds be $a = 0, b = \frac{1}{2}\pi$, we have

$$\left(\int_0^{\frac{\pi}{2}} g(x)\,\mathrm{d}x\right)^2 \leq \frac{\pi}{2}\int_0^{\frac{\pi}{2}} g(x)^2\,\mathrm{d}x.$$

Let $g(x) = \sqrt{\sin x}$, and hence

$$\left(\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,\mathrm{d}x\right)^2 \leq \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x = \frac{\pi}{2}\left[-\cos x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}.$$

Since the integrand $\sqrt{\sin x} \geq 0$ for all $x \in \left[0, \frac{\pi}{2}\right]$, the integral over this interval must be non-negative, and hence

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,\mathrm{d}x \leq \sqrt{\frac{\pi}{2}}.$$

For the left-half of the inequality, consider $g(x) = \sqrt[4]{\sin x}$, and $f(x) = \cos x$ (with the same bounds, $a = 0, b = \frac{1}{2}\pi$). We notice that

$$\begin{aligned}
\int_a^b f(x)g(x)\,\mathrm{d}x &= \int_0^{\frac{1}{2}\pi} \cos x \sqrt[4]{\sin x}\,\mathrm{d}x \\
&= \frac{4}{5}\left[(\sin x)^{\frac{5}{4}}\right]_0^{\frac{1}{2}} \\
&= \frac{4}{5}\left[1^{\frac{5}{4}} - 0^{\frac{5}{4}}\right] \\
&= \frac{4}{5},
\end{aligned}$$

and that

$$\begin{aligned}
\int_a^b f(x)^2\,\mathrm{d}x &= \int_0^{\frac{1}{2}\pi} \cos^2 x\,\mathrm{d}x \\
&= \int_0^{\frac{1}{2}\pi} \frac{1+\cos 2x}{2}\,\mathrm{d}x \\
&= \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{1}{2}\pi} \\
&= \left[\frac{1}{2}\cdot\frac{1}{2}\pi + \frac{1}{4}\sin\pi\right] - \left[\frac{1}{2}\cdot 0 + \frac{1}{4}\sin 0\right] \\
&= \frac{1}{4}\pi.
\end{aligned}$$

Hence, by the Schwarz inequality, we have

$$\frac{16}{25} \leq \frac{1}{4}\pi \cdot \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,\mathrm{d}x,$$

and hence

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,\mathrm{d}x \geq \frac{64}{25\pi}.$$

Combining both sides of the equality, we hence have

$$\frac{64}{25\pi} \leq \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \leq \sqrt{\frac{\pi}{2}},$$

as desired.

## 2017.2 Question 5

1. By taking derivatives with respect to $t$, we have
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2at,$$
and
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2a,$$
hence
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2at} = \frac{1}{t}.$$
The gradient of the normal will hence be $-t$, and hence the normal through $P(ap^2, 2ap)$ will be
$$y - 2ap = -p(x - ap^2).$$
The point $N(an^2, 2an)$ is also on this line, and hence
$$2a(n-p) = -ap(n-p)(n+p).$$
Since $n \neq p$, we must have
$$2 = -p(n+p).$$
Given $p \neq 0$, we have
$$n + p = -\frac{2}{p},$$
and hence
$$n = -p - \frac{2}{p} = -\left(p + \frac{2}{p}\right).$$

2. The distance between $P(ap^2, 2ap)$ and $N(an^2, 2an)$ is given by
$$\begin{aligned}|PN|^2 &= (2ap - 2an)^2 + \left(ap^2 - an^2\right)^2 \\ &= a^2\left[4(p-n)^2 + (p-n)^2(p+n)^2\right] \\ &= a^2(p-n)^2\left[4 + 4\left(-\frac{2}{p}\right)^2\right] \\ &= a^2\left[p + \left(p + \frac{2}{p}\right)\right]^2 \cdot 4\left(\frac{p^2+1}{p^2}\right) \\ &= 4a^2 \cdot 4 \cdot \frac{(p^2+1)^2}{p^2} \cdot \frac{p^2+1}{p^2} \\ &= 16a^2\frac{(p^2+1)^3}{p^4}.\end{aligned}$$
Let $f(p) = \frac{(p^2+1)^3}{p^4}$. By differentiation,
$$\begin{aligned}f'(p) &= \frac{3 \cdot 2p \cdot (p^2+1)^2 \cdot p^4 - (p^2+1)^3 \cdot 4 \cdot p^3}{p^8} \\ &= \frac{2(p^2+1)^2p^3}{p^8}\left[3p^2 - 2(p^2+1)\right] \\ &= \frac{2(p^2+1)^2}{p^5}\left(p^2 - 2\right).\end{aligned}$$
This means that $f'(p) = 0$ precisely when $p^2 - 2 = 0$, i.e. $p = \pm\sqrt{2}$.
When $0 < p < \sqrt{2}$, $f'(p) < 0$, and when $\sqrt{2} < p$, $f'(p) > 0$.
When $p < -\sqrt{2}$, $f'(p) < 0$, and when $-\sqrt{2} < p < 0$, $f'(p) > 0$.
This means that when $p^2 - 2 = 0$ (i.e. $p = \pm\sqrt{2}$), $f(p)$ has a minimum.
Since $|PN|^2 = \frac{16}{a^2}f(p)$ is a positive multiple of $f(p)$, we must have that $|PN|^2$ is minimised when $p^2 = 2$.

3. Since $Q(aq^2, 2aq)$ is on the circle with diameter $PN$, we must have that $QP$ and $QN$ are perpendicular.

   The gradient of $QP$ is given by

$$m_{QP} = \frac{2aq - 2ap}{aq^2 - ap^2} = \frac{2(q-p)}{(q+p)(q-p)} = \frac{2}{q+p},$$

   and the gradient of $QN$ is given by

$$m_{QN} = \frac{2aq - 2an}{aq^2 - an^2} = \frac{2(q-n)}{(q+n)(q-n)} = \frac{2}{q+n}.$$

   Since $QP$ and $QN$ are perpendicular, we must have

$$\begin{aligned} m_{QP} \cdot m_{QN} = -1 &\iff \frac{2}{q+p} \cdot \frac{2}{q+n} = -1 \\ &\iff -4 = (q+p)(q+n) \\ &\iff q^2 + (p+n)q + pn = -4 \\ &\iff q^2 - \frac{2}{p} \cdot q - p^2 - 2 = -4 \\ &\iff p^2 - q^2 + \frac{2q}{p} = 2, \end{aligned}$$

   as desired.

   When $|PN|$ is a minimum, we have $p = \pm\sqrt{2}$, and hence

$$2 - q^2 \pm \sqrt{2}q = 2,$$

   which gives

$$q(q \mp \sqrt{2}) = 0.$$

   Hence, $q = 0$, or $q = \pm\sqrt{2}$ (which means $p = q$, which cannot be the case). When $q = 0$, $Q(0, 0)$ is at the origin, as desired.

## 2017.2 Question 6

1. We first look at the base case where $n = 1$. $S_1 = \frac{1}{1} = 1$, and $2 \cdot \sqrt{1} - 1 = 1$, so

$$S_1 \leq 2 \cdot \sqrt{1} - 1$$

holds, and the original statement holds for when $n = 1$.

Assume this holds for some $n = k \in \mathbb{N}$, i.e., $S_k \leq 2\sqrt{k} - 1$. We have

$$\begin{aligned} S_{k+1} &= \sum_{r=1}^{k+1} \frac{1}{\sqrt{r}} \\ &= \sum_{r=1}^{k} \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{k+1}} \\ &= S_k + \frac{1}{\sqrt{k+1}} \\ &\leq 2\sqrt{k} + \frac{1}{\sqrt{k+1}} - 1. \end{aligned}$$

We would like to show

$$2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}.$$

Notice that

$$\begin{aligned} 2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} &\iff 2\sqrt{k(k+1)} + 1 \leq 2(k+1) \\ &\iff 2\sqrt{k(k+1)} \leq 2k+1 \\ &\iff 4k(k+1) \leq (2k+1)^2 \\ &\iff 4k^2 + 4k \leq 4k^2 + 4k + 1 \\ &\iff 0 \leq 1, \end{aligned}$$

which is true.

Hence,

$$S_{k+1} \leq 2\sqrt{k} + \frac{1}{\sqrt{k+1}} - 1 \leq 2\sqrt{k+1} - 1,$$

which is precisely the statement for $n = k + 1$.

The original statement holds for the base case where $n = 0$, and assuming it holds for some $n = k \in \mathbb{N}$, it holds for $n = k + 1$. Hence, by the principle of mathematical induction, the original statement holds for all $n \in \mathbb{N}$.

2. For $k \geq 0$, we notice

$$\begin{aligned} (4k+1)\sqrt{k+1} > (4k+3)\sqrt{k} &\iff (4k+1)^2(k+1) > (4k+3)^2 k \\ &\iff (16k^2 + 8k + 1)(k+1) > (16k^2 + 24k + 9)k \\ &\iff 16k^3 + 8k^2 + k + 16k^2 + 8k + 1 > 16k^3 + 24k^2 + 9k \\ &\iff 1 > 0, \end{aligned}$$

which is true.

We claim that $C = \frac{3}{2}$ is the smallest number $C$ which makes this true. We first show that $C = \frac{3}{2}$ makes the statement true by induction. For the base case where $n = 1$, $S_1 = 1$, and

$$2\sqrt{1} + \frac{1}{2\sqrt{1}} - \frac{3}{2} = \frac{5}{2} - \frac{3}{2} = 1,$$

and so this statement holds for $n = 1$.

Now, assume that this statement holds for some $n = k \in \mathbb{N}$, i.e.

$$S_k \geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} - C.$$

We have

$$\begin{aligned} S_{k+1} &= S_k + \frac{1}{\sqrt{k+1}} \\ &\geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - C. \end{aligned}$$

We would like to show that

$$2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}}.$$

Notice that

$$\begin{aligned} & 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} \\ \iff & 2\sqrt{k} + \frac{1}{2\sqrt{k}} \geq 2\sqrt{k+1} - \frac{1}{2\sqrt{k+1}} \\ \iff & \frac{4k+1}{2\sqrt{k}} \geq \frac{4(k+1)-1}{2\sqrt{k+1}} \\ \iff & (4k+1)\sqrt{k+1} \geq (4k+3)\sqrt{k}, \end{aligned}$$

which is implied by the proven inequality, and hence

$$S_{k+1} \geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - C \geq 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} - C,$$

which precisely proves the statement for $n = k + 1$.

The claimed statement holds for the base case where $n = 1$, and given it holds for some $n = k \in \mathbb{N}$, it holds for $n = k + 1$. Hence, the statement holds for all $n \in \mathbb{N}$ when $C = \frac{3}{2}$.

If $C < \frac{3}{2}$, we have for $n = 1$

$$2\sqrt{1} + \frac{1}{2\sqrt{1}} - C > \frac{5}{2} - \frac{3}{2} = 1,$$

but

$$S_1 = 1,$$

so the statement does not hold for when $n = 1$.

Hence, the smallest number $C$ for the statement to be true is $C = \frac{3}{2}$.

## 2017.2 Question 7

1. Since ln is an increasing function, for $0 < x < 1$, we have $\ln x < 0$, and

$$\begin{aligned} f(x) > x &\iff \ln f(x) > \ln x \\ &\iff \ln x^x > \ln x \\ &\iff x \ln x > \ln x \\ &\iff x < 1, \end{aligned}$$

which is true since $0 < x < 1$.

Notice that

$$\begin{aligned} x < g(x) < f(x) &\iff \ln x < \ln x^{f(x)} < \ln x^x \\ &\iff \ln x < x^x \ln x < x \ln x \\ &\iff 1 > x^x > x. \end{aligned}$$

The right inequality is shown by the previous part. For the left inequality, we have

$$\begin{aligned} 1 > x^x &\iff \ln 1 > x \ln x \\ &\iff 0 > x \ln x \end{aligned}$$

must be true, since $0 < x < 1$ and $\ln x < 0$.

Hence, we have $x < g(x) < f(x)$ for $0 < x < 1$.

When $x > 1$, we claim that $x < f(x) < g(x)$.

2. Notice that

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} x^x \\ &= \frac{\mathrm{d}}{\mathrm{d}x} \exp(x \ln x) \\ &= \exp(x \ln x) \cdot \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) \\ &= \exp(x \ln x) \cdot (\ln x + 1) \\ &= f(x) \cdot (\ln x + 1). \end{aligned}$$

$f'(x) = 0$ if and only if $\ln x + 1 = 0$, which holds if and only if $x = \frac{1}{e}$.

3. We have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \exp(x \ln x) = \exp(0) = 1,$$

and hence

$$\lim_{x \to 0} g(x) = 0.$$

4. Let $h(x) = \frac{1}{x} + \ln x$. We have

$$h'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{x-1}{x^2}.$$

When $0 < x < 1$, $h'(x) < 0$, and when $1 < x$, $h'(x) > 0$. Hence, $h$ takes a minimum when $x = 1$, and $h(1) = \frac{1}{1} + \ln 1 = 1$.

This shows precisely that

$$\frac{1}{x} + \ln x \geq 1$$

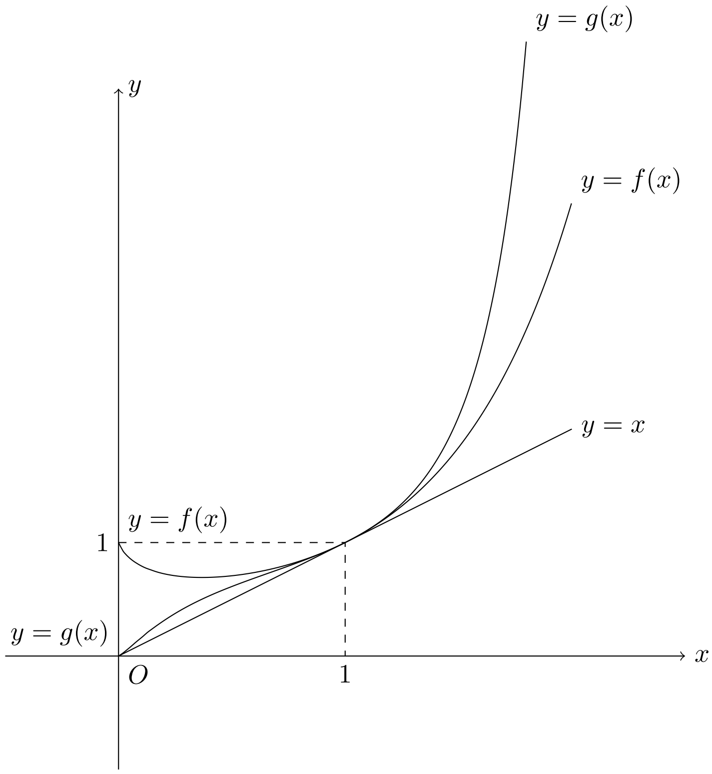
for $x > 0$.

Notice that

$$\begin{aligned}
g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} x^{f(x)} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \exp(f(x)\ln x) \\
&= \exp(f(x)\ln x) \cdot \left(\frac{1}{x} \cdot f(x) + f'(x)\ln x\right) \\
&= g(x) \cdot \left(\frac{1}{x} \cdot f(x) + f(x) \cdot (\ln x + 1) \cdot \ln x\right) \\
&= f(x)g(x) \cdot \left(\frac{1}{x} + \ln x(\ln x + 1)\right) \\
&\geq f(x)g(x) \cdot \left(1 + (\ln x)^2\right) \\
&> 0,
\end{aligned}$$

since $f(x), g(x) > 0$ for $x > 0$ (since they are both exponentials), and $1 + (\ln x)^2 \geq 1 > 0$ as well.

The graphs of the functions look as follows.



## 2017.2 Question 8

The line through $A$ perpendicular to $BC$ is

$$l_1 : \mathbf{r} = \mathbf{a} + \lambda \mathbf{u}, \lambda \in \mathbb{R}.$$

The line through $B$ perpendicular to $CA$ is

$$l_2 : \mathbf{r} = \mathbf{b} + \mu \mathbf{v}, \mu \in \mathbb{R}.$$

Since $P$ is the intersection of $l_1$ and $l_2$, we must have

$$\mathbf{a} + \lambda \mathbf{u} = \mathbf{b} + \mu \mathbf{v},$$

and hence solving for $\mathbf{v}$ we have

$$\mathbf{v} = \frac{1}{\mu} \left( \mathbf{a} + \lambda \mathbf{u} - \mathbf{b} \right).$$

Since $\mathbf{v}$ is perpendicular to $CA$, we must have $\mathbf{v} \cdot (\mathbf{a} - \mathbf{c}) = 0$, and hence

$$\begin{aligned} & \frac{1}{\mu} \left( \mathbf{a} + \lambda \mathbf{u} - \mathbf{b} \right) \cdot (\mathbf{a} - \mathbf{c}) = 0 \\ \iff & (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c}) + \lambda \mathbf{u} \cdot (\mathbf{a} - \mathbf{c}) = 0 \\ \iff & \lambda = -\frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c})}{\mathbf{u} \cdot (\mathbf{a} - \mathbf{c})}. \end{aligned}$$

Hence, the position vector of $P$, $\mathbf{p}$, must satisfy that

$$\mathbf{p} = \mathbf{a} + \lambda \mathbf{u} = \mathbf{a} - \frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c})}{\mathbf{u} \cdot (\mathbf{a} - \mathbf{c})} \mathbf{u}.$$

$CP$ is perpendicular to $AB$ if and only if $(\mathbf{p} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) = 0$. We notice

$$\begin{aligned} (\mathbf{p} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) &= (\mathbf{a} + \lambda \mathbf{u} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) + \lambda \mathbf{u} \left( \mathbf{b} - \mathbf{a} \right). \end{aligned}$$

Since $\mathbf{u}$ is perpendicular to $BC$, we must have $\mathbf{u} \cdot (\mathbf{c} - \mathbf{b})$, and hence $\mathbf{u} \cdot \mathbf{c} = \mathbf{u} \cdot \mathbf{b}$. Hence,

$$\begin{aligned} (\mathbf{p} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) + \lambda \mathbf{u} \cdot (\mathbf{b} - \mathbf{a}) \\ &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) + \lambda \mathbf{u} \cdot (\mathbf{c} - \mathbf{a}) \\ &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) - \frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c})}{\mathbf{u} \cdot (\mathbf{a} - \mathbf{c})} \mathbf{u} \cdot (\mathbf{c} - \mathbf{a}) \\ &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) + (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c}) \\ &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) - (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= 0, \end{aligned}$$

and hence $CP$ is perpendicular to $AB$.

## 2017.2 Question 12

1. Let $X \sim \mathrm{Po}(\lambda)$ and $Y \sim \mathrm{Po}(\mu)$. $X$ and $Y$ take values of non-negative integers. Hence, for any non-negative integer $r$, we have

$$\begin{aligned}
\mathrm{P}(X+Y=r) &= \sum_{t=0}^{r} \mathrm{P}(X=t, Y=r-t) \\
&= \sum_{t=0}^{r} \mathrm{P}(X=t)\,\mathrm{P}(Y=r-t) \\
&= \sum_{t=0}^{r} \frac{\lambda^t}{e^\lambda \cdot t!} \cdot \frac{\mu^{r-t}}{e^\mu \cdot (r-t)!} \\
&= \frac{1}{e^{\lambda+\mu}} \cdot \sum_{t=0}^{r} \frac{\lambda^t \mu^{r-t}}{t!(r-t)!} \\
&= \frac{1}{e^{\lambda+\mu} r!} \cdot \sum_{t=0}^{r} \frac{r!\lambda^t \mu^{r-t}}{t!(r-t)!} \\
&= \frac{1}{e^{\lambda+\mu} r!} \cdot \sum_{t=0}^{r} \binom{r}{t} \lambda^t \mu^{r-t} \\
&= \frac{1}{e^{\lambda+\mu} r!} (\lambda+\mu)^r \\
&= \frac{(\lambda+\mu)^r}{e^{\lambda+\mu} r!},
\end{aligned}$$

   which is precisely the probability mass function for $\mathrm{Po}(\lambda+\mu)$, and hence $X+Y \sim \mathrm{Po}(\lambda+\mu)$.

2. We consider the probability mass function for the number of fishes Adam has caught in this situation. Given $X+Y=k$, the only values that $X$ can take are $0, 1, \cdots, k$, and hence consider $x = 0, 1, \cdots, k$, we have

$$\begin{aligned}
\mathrm{P}(X=x \mid X+Y=k) &= \frac{\mathrm{P}(X=x, X+Y=k)}{\mathrm{P}(X+Y=k)} \\
&= \frac{\mathrm{P}(X=x, Y=k-x)}{\mathrm{P}(X+Y=k)} \\
&= \frac{\mathrm{P}(X=x) \cdot \mathrm{P}(Y=k-x)}{\mathrm{P}(X+Y=k)} \\
&= \frac{\frac{\lambda^x}{e^\lambda x!} \cdot \frac{\mu^{k-x}}{e^\mu (k-x)!}}{\frac{(\lambda+\mu)^k}{e^{\lambda+\mu} k!}} \\
&= \frac{\lambda^x \mu^{k-x}}{(\lambda+\mu)^k} \cdot \frac{k!}{x!(k-x)!} \\
&= \binom{k}{x} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^x \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{k-x}.
\end{aligned}$$

   This is precisely the probability mass function for the binomial distribution $\mathrm{B}\left(k, \frac{\lambda}{\lambda+\mu}\right)$, and we can say that

$$(X \mid X+Y=k) \sim \mathrm{B}\left(k, \frac{\lambda}{\lambda+\mu}\right).$$

3. When the first fish is caught, this is $X+Y=1$, and $X=1$. Hence, the probability is

$$\mathrm{P}(X=1 \mid X+Y=1) = \binom{1}{1} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^1 \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{1-1} = \frac{\lambda}{\lambda+\mu}.$$

4. There is a probability of $\frac{\lambda}{\lambda+\mu}$ of Adam catching the first fish, and in this case the waiting time is first for the fish to come up (which is $\frac{1}{\lambda+\mu}$), plus the waiting time of Eve's fish to come up (which is $\frac{1}{\mu}$), summed together. This applies the other way around as well if Eve catches the first fish.

Hence, the expected time is

$$\begin{aligned}
&\frac{\lambda}{\lambda+\mu}\cdot\left(\frac{1}{\lambda+\mu}+\frac{1}{\mu}\right)+\frac{\mu}{\lambda+\mu}\cdot\left(\frac{1}{\lambda+\mu}+\frac{1}{\lambda}\right)\\
&=\frac{1}{\lambda+\mu}\cdot\left(\frac{\lambda}{\lambda+\mu}+\frac{\lambda}{\mu}+\frac{\mu}{\lambda+\mu}+\frac{\mu}{\lambda}\right)\\
&=\frac{1}{\lambda+\mu}\cdot\left(1+\frac{\lambda^2+\mu^2}{\lambda\mu}\right)\\
&=\frac{\lambda^2+\lambda\mu+\mu^2}{\lambda\mu(\lambda+\mu)}.
\end{aligned}$$

## 2017.2 Question 13

1. For each try, there is a probability of $\frac{1}{n}$ of getting the correct key, and $1-\frac{1}{n}$ otherwise. Let $X_1$ denote the number of attempts to open the door, we can see that $X_1 \sim \text{Geo}\left(\frac{1}{n}\right)$, and hence using the formula for a geometric distribution,

$$\mathrm{E}(X_1) = n.$$

The way to consider the binomial expansion is as follows. First, note the probability mass function of $X_1$ is

$$\mathrm{P}(X_1 = x) = \left(1-\frac{1}{n}\right)^{x-1} \cdot \frac{1}{n},$$

and hence the expectation is given by

$$\begin{aligned}\mathrm{E}(X_1) &= \sum_{x=1}^{\infty} x\,\mathrm{P}(X_1 = x) \\ &= \sum_{x=1}^{\infty} x \cdot \left(1-\frac{1}{n}\right)^{x-1} \cdot \frac{1}{n} \\ &= \frac{1}{n} \cdot \sum_{x=1}^{\infty} x \cdot \left(1-\frac{1}{n}\right)^{x-1}.\end{aligned}$$

Consider the binomial expansion of $(1-q)^{-2}$. We have

$$\begin{aligned}(1-q)^{-2} &= \sum_{t=0}^{\infty} \frac{(-q)^t \cdot \prod_{r=1}^{t}(-2+1-t)}{t!} \\ &= \sum_{t=0}^{\infty} \frac{(-1)^t q^t (-1)^t \prod_{r=1}^{t}(1+t)}{t!} \\ &= \sum_{t=0}^{\infty} \frac{q^t (t+1)!}{t!} \\ &= \sum_{t=0}^{\infty} (t+1) q^t.\end{aligned}$$

Let $q = 1 - \frac{1}{n}$. We can see

$$\begin{aligned}\mathrm{E}(X_1) &= \frac{1}{n} \cdot \sum_{x=1}^{\infty} x \cdot \left(1-\frac{1}{n}\right)^{x-1} \\ &= \frac{1}{n} \cdot \sum_{x=0}^{\infty} (x+1) \cdot q^x \\ &= \frac{1}{n} \cdot (1-q)^{-2} \\ &= \frac{1}{n} \cdot \left(\frac{1}{n}\right)^{-2} \\ &= n,\end{aligned}$$

precisely what we had before.

2. Let $X_2$ be the number of attempts to open the door in this case. Considering the probability mass

function of $X_2$, we have for $x = 1, 2, \ldots, n$, that

$$\begin{aligned}
\mathrm{P}(X_2 = x) &= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{n-(x-2)-1}{n-(x-2)} \cdot \frac{1}{n-(x-1)} \\
&= \frac{(n-1)!/(n-x)!}{n!/(n-x)!} \\
&= \frac{(n-1)!}{n!} \\
&= \frac{1}{n}.
\end{aligned}$$

This shows that $X_2$ follows a discrete uniform distribution on $\{1, 2, \ldots, n\}$, i.e., $X_2 \sim \mathrm{U}(n)$. Hence, $\mathrm{E}(X_2) = \frac{n+1}{2}$.

3. Let $X_3$ be the number of attempts to open the door in this case. Considering the probability mass function of $X_2$, we have for $x = 1, 2, \ldots$, that

$$\begin{aligned}
\mathrm{P}(X_3 = x) &= \frac{n-1}{n} \cdot \frac{n}{n+1} \cdots \frac{n+x-3}{n+x-2} \cdot \frac{1}{n+x-1} \\
&= \frac{(n+x-3)!/(n-2)!}{(n+x-1)!/(n-1)!} \\
&= \frac{(n+x-3)!(n-1)!}{(n+x-1)!(n-2)!} \\
&= \frac{n-1}{(n+x-1)(n+x-2)},
\end{aligned}$$

which is precisely what is desired.

By partial fractions, we have

$$\mathrm{P}(X_3 = x) = (n-1) \cdot \left( \frac{2}{n+x-2} - \frac{1}{n+x-1} \right),$$

and hence the expected number of attempts is

$$\begin{aligned}
\mathrm{E}(X_3) &= \sum_{x=1}^{\infty} (n-1) \cdot x \cdot \left( \frac{1}{n+x-2} - \frac{1}{n+x-1} \right) \\
&= (n-1) \sum_{x=1}^{\infty} x \left( \frac{1}{n+x-2} - \frac{1}{n+x-1} \right).
\end{aligned}$$

We consider the partial sum of this infinite sum op to $x = t$, and

$$\begin{aligned}
\sum_{x=1}^{t} x \left( \frac{1}{n+x-2} - \frac{1}{n+x-1} \right) &= \sum_{x=1}^{t} \frac{x}{n+x-2} - \sum_{x=1}^{t} \frac{x}{n+x-1} \\
&= \sum_{x=0}^{t-1} \frac{x+1}{n+x-1} - \sum_{x=1}^{t} \frac{x}{n+x-1} \\
&= \frac{1}{n-1} + \sum_{x=1}^{t-1} \frac{1}{n+x-1} - \frac{t}{n+t-1} \\
&= \sum_{x=0}^{t-1} \frac{1}{n+x-1} - \frac{t}{n+t-1} \\
&= \sum_{x=n-1}^{n+t-2} \frac{1}{x} - \frac{t}{n+t-1}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathrm{E}(X_3) &= (n-1)\sum_{x=1}^{\infty} x\left(\frac{1}{n+x-2}-\frac{1}{n+x-1}\right) \\
&= (n-1)\lim_{t\to\infty}\left(\sum_{x=n-1}^{n+t-2}\frac{1}{x}-\frac{t}{n+t-1}\right) \\
&= (n-1)\lim_{t\to\infty}\left(\sum_{x=1}^{n+t-2}\frac{1}{x}-\sum_{x=1}^{n-2}\frac{1}{x}-\frac{t}{n+t-1}\right) \\
&= (n-1)\left(\sum_{x=1}^{\infty}\frac{1}{x}-\sum_{x=1}^{n-2}\frac{1}{x}-1\right)
\end{aligned}$$

does not converge since the first term (harmonic sum) diverges, and the rest of the terms are finite.