

2016 Paper 3

| | | |
|-----------|-------------|----|
| 2016.3.1 | Question 1 | 7 |
| 2016.3.2 | Question 2 | 10 |
| 2016.3.3 | Question 3 | 12 |
| 2016.3.4 | Question 4 | 14 |
| 2016.3.5 | Question 5 | 16 |
| 2016.3.6 | Question 6 | 18 |
| 2016.3.7 | Question 7 | 20 |
| 2016.3.8 | Question 8 | 22 |
| 2016.3.12 | Question 12 | 25 |
| 2016.3.13 | Question 13 | 26 |

2016.3 Question 1

Notice that

$$I_n = \int_{-\infty}^{+\infty} \frac{dx}{(x^2 + 2ax + b)^n} = \int_{-\infty}^{+\infty} \frac{dx}{((x+a)^2 + (b-a^2))^n}.$$

1. Let $x + a = \sqrt{b - a^2} \tan u$. When $x \rightarrow -\infty$, $u \rightarrow -\frac{\pi}{2}$, and when $x \rightarrow +\infty$, $u \rightarrow \frac{\pi}{2}$. We have also

$$\begin{aligned} dx &= d(x + a) = d\sqrt{b - a^2} \tan u \\ &= \sqrt{b - a^2} d \tan u \\ &= \sqrt{b - a^2} \sec^2 u du. \end{aligned}$$

Therefore, we have

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} \frac{dx}{(x+a)^2 + (b-a^2)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2} \sec^2 u du}{(\sqrt{b-a^2} \tan u)^2 + (b-a^2)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2} \sec^2 u du}{(b-a^2)(\tan^2 u + 1)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2 u du}{\sqrt{b-a^2} \sec^2 u} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{\sqrt{b-a^2}} \\ &= \frac{\pi}{\sqrt{b-a^2}}, \end{aligned}$$

as desired.

2. Using the same substitution, we have

$$\begin{aligned} I_n &= \int_{-\infty}^{+\infty} \frac{dx}{[(x+a)^2 + (b-a^2)]^n} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2} \sec^2 u du}{[(b-a^2) \sec^2 u]^n} \\ &= \frac{1}{\sqrt{b-a^2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{[(b-a^2) \sec^2 u]^{n-1}}. \end{aligned}$$

Therefore,

$$2n(b-a^2)I_{n+1} = (2n-1)I_n,$$

is equivalent to

$$2n\sqrt{b-a^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{[(b-a^2) \sec^2 u]^n} = (2n-1) \frac{1}{\sqrt{b-a^2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{[(b-a^2) \sec^2 u]^{n-1}}$$

is equivalent to

$$2n(b-a^2) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{[(b-a^2) \sec^2 u]^n} = (2n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{[(b-a^2) \sec^2 u]^{n-1}}$$

is equivalent to

$$2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{\sec^{2n} u} = (2n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{\sec^{2n-2} u}.$$

Notice that

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{\sec^{2n-2} u} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2 u \, du}{\sec^{2n} u} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d \tan u}{\sec^{2n} u} \\
&= \lim_{\substack{a \rightarrow \frac{\pi}{2} \\ b \rightarrow -\frac{\pi}{2}}} \left[\frac{\tan u}{\sec^{2n} u} \right]_b^a - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan u \, d \sec^{-2n} u \\
&= \lim_{\substack{a \rightarrow \frac{\pi}{2} \\ b \rightarrow -\frac{\pi}{2}}} [\sin u \cos^{2n-1} u]_b^a - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -2n \sec u \tan u \sec^{-2n-1} u \tan u \, du \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tan^2 u \, du}{\sec^{2n} u} \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\sec^2 u - 1) \, du}{\sec^{2n} u} \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{\sec^{2n-2} u} - 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{\sec^{2n} u}.
\end{aligned}$$

This means

$$(2n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{\sec^{2n-2} u} = 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{\sec^{2n} u},$$

which is exactly what was desired.

3. Proof by induction:

- **Base Case.** When $n = 1$,

$$\text{LHS} = I_1 = \frac{\pi}{\sqrt{b-a^2}},$$

$$\text{RHS} = \frac{\pi}{2^{2 \cdot 1 - 2} (b-a^2)^{1 - \frac{1}{2}}} \binom{2 \cdot 1 - 2}{1 - 1} = \frac{\pi}{\sqrt{b-a^2}} \binom{0}{0} = \frac{\pi}{\sqrt{b-a^2}}.$$

- **Induction Hypothesis.** Assume for some $n = k \in \mathbb{N}$, we have

$$I_n = \frac{\pi}{2^{2n-2} (b-a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1}.$$

- **Induction Step.** When $n = k + 1$,

$$\begin{aligned}
I_n &= I_{k+1} \\
&= \frac{2k+1}{2(k+1)(b-a^2)} I_k \\
&= \frac{2k+1}{2(k+1)(b-a^2)} \cdot \frac{\pi}{2^{2k-2} (b-a^2)^{k-\frac{1}{2}}} \binom{2k-2}{k-1} \\
&= \frac{\pi}{2^{2k} (b-a^2)^{k+\frac{1}{2}}} \frac{(2k-2)!}{(k-1)!(k-1)!} \frac{(2k+1)(2k+2)}{(k+1)^2} \\
&= \frac{\pi}{2^{2k} (b-a^2)^{k+\frac{1}{2}}} \frac{2k!}{k!k!} \\
&= \frac{\pi}{2^{2k} (b-a^2)^{k+\frac{1}{2}}} \binom{2k}{k} \\
&= \frac{\pi}{2^{2n-2} (b-a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1}.
\end{aligned}$$

Therefore, by the principle of mathematical induction, for $n \in \mathbb{N}$,

$$I_n = \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1},$$

as desired.

2016.3 Question 2

1. For $y^2 = 4ax$, we have $x = \frac{y^2}{4a}$, and therefore

$$\frac{dx}{dy} = \frac{2y}{4a}.$$

Therefore, the normal through Q , l_Q satisfies that

$$l_Q : x - aq^2 = -\frac{4a}{2 \cdot 2aq} \cdot (y - 2aq),$$

i.e.

$$l_Q : q(x - aq^2) = -(y - 2aq).$$

Since $P \in l_Q$, we must have

$$\begin{aligned} q(ap^2 - aq^2) &= -(2ap - 2aq) \\ aq(p + q)(p - q) &= -2a(p - q) \\ pq + q^2 &= -2 \\ q^2 + pq + 2 &= 0 \end{aligned}$$

as desired.

2. We also have

$$r^2 + pr + 2 = 0.$$

Since $q \neq r$, q, r are the solutions to the equation

$$x^2 + px + 2 = 0,$$

and therefore $q + r = -p$, $qr = 2$.

Note that the equation for QR satisfies that

$$m_{QR} = \frac{2ar - 2aq}{ar^2 - aq^2} = \frac{2}{r + q}.$$

Therefore, l_{QR} satisfies that

$$\begin{aligned} l_{QR} : y - 2aq &= \frac{2}{r + q}(x - aq^2) \\ y &= \frac{2}{r + q} \left(x - aq^2 + \frac{r + q}{2} \cdot 2aq \right) \\ y &= \frac{2}{r + q} (x - aq^2 + aq^2 + aqr) \\ y &= \frac{2}{r + q} (x + aqr) \\ y &= -\frac{2}{p}(x + 2a). \end{aligned}$$

This passes through a fixed point $(-2a, 0)$.

3. OP has equation $y = \frac{2ap}{ap^2}x$, which is $y = \frac{2x}{p}$. Therefore, since $T = OP \cap QR$, x_T must satisfy that

$$\begin{aligned} -\frac{2}{p}(x + 2a) &= \frac{2x}{p}, \\ -(x + 2a) &= x \\ x &= -a. \end{aligned}$$

Therefore, $y_T = -\frac{2a}{p}$, $T \left(-a, -\frac{2a}{p} \right)$ lies on the line $x = -a$ which is independent of p .

The distance from the x -axis to T is $\left| \frac{2a}{p} \right| = \frac{2a}{|p|}$.

Notice that since $qr = 2$, q and r must take the same parity, and therefore $|p| = |q| + |r|$. By the AM-GM inequality, we have

$$|q| + |r| \geq 2\sqrt{|q| \cdot |r|} = 2\sqrt{2},$$

with the equal sign holding if and only if $|q| = |r|$, $q = r$, which is impossible.

Therefore, $|p| > 2\sqrt{2}$ and therefore $\frac{2a}{|p|} < \sqrt{2}$ as desired.

2016.3 Question 3

1. We have that

$$\begin{aligned} \frac{d}{dx} \frac{e^x P(x)}{Q(x)} &= \frac{Q(x) [e^x P'(x) + e^x P(x)] - Q'(x) e^x P(x)}{Q(x)^2} \\ &= e^x \frac{[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)]}{Q(x)^2} \\ &= e^x \frac{x^3 - 2}{(x+1)^2}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)]}{Q(x)^2} &= \frac{x^3 - 2}{(x+1)^2} \\ (x+1)^2 [Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)] &= Q(x)^2 (x^3 - 2). \end{aligned}$$

If we plug in $x = -1$ on both sides, we have LHS = 0 and RHS = $Q(-1)^2 \cdot (-3)$.

Therefore, $Q(-1)^2 = 0$, $Q(-1) = 0$.

Since $Q(x) \in \mathbb{P}[x]$, we must have

$$(x+1) \mid Q(x)$$

as desired.

Therefore, $\deg Q \geq 1$, $\deg \text{RHS} = 3 + 2 \deg Q$.

If $\deg P = -\infty$, $P(x) = 0$, LHS = 0 which is impossible.

If $\deg P = 0$, $P(x) = C \in \mathbb{R} \setminus \{0\}$, LHS = $C(x+1)^2 Q(x)$, $\deg \text{LHS} = \deg q + 2$, which is impossible.

Therefore, we have $\deg P' = \deg P - 1$. Hence,

$$\deg Q(x)P'(x) = \deg P'(x)Q(x) = \deg P + \deg Q - 1,$$

and

$$\deg Q(x)P(x) = \deg P + \deg Q.$$

Therefore,

$$\deg \text{LHS} = 2 + \deg P + \deg Q = \deg \text{RHS},$$

which gives

$$\deg P = \deg Q + 1,$$

as desired.

When $Q(x) = x + 1$, let $P(x) = ax^2 + bx + c$ where $a \neq 0$. We have $P'(x) = 2ax + b$. Therefore,

$$\begin{aligned} (x+1)^2 [Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)] &= Q(x)^2 (x^3 - 2) \\ Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x) &= x^3 - 2 \\ (x+1)(2ax+b) + (x+1)(ax^2+bx+c) - (ax^2+bx+c) &= x^3 - 2 \\ (x+1)(2ax+b) + x(ax^2+bx+c) &= x^3 - 2 \\ ax^3 + (2a+b)x^2 + (2a+b+c)x + b &= x^3 - 2. \end{aligned}$$

This solves to $(a, b, c) = (1, -2, 0)$. Therefore, $P(x) = x^2 - 2x$.

2. In this case, we must have that

$$(x+1) [Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)] = Q(x)^2.$$

Therefore, $Q(x) = (x+1)R(x)$ for some $R(x) \in \mathbb{P}[x]$. We may assume $P(-1) \neq 0$.

Hence, $Q'(x) = (x+1)R'(x) + R(x)$

Plugging this in gives us

$$(x+1)R(x)P'(x) + (x+1)R(x)P(x) - [(x+1)R'(x) + R(x)]P(x) = (x+1)R(x)^2,$$

which simplifies to

$$(x+1)[R(x)P'(x) + R(x)P(x) - R'(x)P(x)] - R(x)P(x) = (x+1)R(x)^2.$$

Let $x = -1$, and we can see $x+1$ divides $R(x)$, since $x+1$ can't divide $P(x)$.

Therefore, let $R(x) = (x+1)S(x)$, therefore $R'(x) = S(x) + (x+1)S'(x)$.

This gives

$$(x+1)S(x)[P'(x) + P(x)] - [S(x) + (x+1)S'(x)]P(x) - S(x)P(x) = (x+1)^2S(x)^2,$$

which simplifies to

$$(x+1)[S(x)P'(x) + S(x)P(x) - S'(x)P(x)] - 2S(x)P(x) = (x+1)^2S(x)^2.$$

Therefore, we can see that $x+1$ divides $S(x)$ by similar reasons.

Repeating this, we can conclude that there are arbitrarily many factors of $x+1$ in $Q(x)$ (proof by infinite descent), which is impossible.

Formally speaking, let $Q(x) = (x+1)^n T(x)$ where $T(-1) \neq 0$, $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} Q'(x) &= n(x+1)^{n-1}T(x) + (x+1)^n T'(x) \\ &= (x+1)^{n-1} [nT(x) + (x+1)T'(x)]. \end{aligned}$$

Therefore,

$$(x+1)[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)] = Q(x)^2$$

simplifies to

$$(x+1)^{n+1}T(x)[P'(x) + P(x)] - (x+1)^n [nT(x) + (x+1)T'(x)]P(x) = (x+1)^{2n}T(x)^2,$$

which further simplifies to

$$(x+1)[T(x)P'(x) + T(x)P(x) - T'(x)P(x)] - nT(x)P(x) = (x+1)^n T(x)^2.$$

Now, let $x = -1$, we have that $nT(-1)P(-1) = 0$. But $n \neq 0$, $T(-1) \neq 0$, $P(-1) \neq 0$, which gives a contradiction.

Therefore, such P and Q do not exist.

2016.3 Question 4

1. Notice that

$$\frac{1}{1+x^r} - \frac{1}{1+x^{r+1}} = \frac{x^{r+1} - x^r}{(1+x^r)(1+x^{r+1})} = \frac{x^r(x-1)}{(1+x^r)(1+x^{r+1})}.$$

Therefore, we have

$$\begin{aligned} \sum_{r=1}^N \frac{x^r}{(1+x^r)(1+x^{r+1})} &= \sum_{r=1}^N \frac{1}{x-1} \left[\frac{1}{1+x^r} - \frac{1}{1+x^{r+1}} \right] \\ &= \frac{1}{x-1} \sum_{r=1}^N \left[\frac{1}{1+x^r} - \frac{1}{1+x^{r+1}} \right] \\ &= \frac{1}{x-1} \left[\frac{1}{1+x} - \frac{1}{1+x^{n+1}} \right]. \end{aligned}$$

For $|x| < 1$, as $n \rightarrow \infty$, $x^{n+1} \rightarrow 0$. Therefore,

$$\begin{aligned} \sum_{r=1}^{\infty} \frac{x^r}{(1+x^r)(1+x^{r+1})} &= \frac{1}{x-1} \left[\frac{1}{1+x} - 1 \right] \\ &= \frac{1}{x-1} \cdot \frac{-x}{1+x} \\ &= \frac{x}{1-x^2} \end{aligned}$$

as desired.

2. Notice that

$$\begin{aligned} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) &= \frac{2}{e^{ry} + e^{-ry}} \cdot \frac{2}{e^{(r+1)y} + e^{-(r+1)y}} \\ &= \frac{4e^{-ry-(r+1)y}}{(1+e^{-2ry})(1+e^{-2(r+1)y})} \\ &= 4e^{-y} \frac{e^{-2ry}}{(1+e^{-2ry})(1+e^{-2(r+1)y})}. \end{aligned}$$

Let $x = e^{-2y}$. We have

$$\operatorname{sech}(ry) \operatorname{sech}((r+1)y) = 4e^{-y} \frac{x^r}{(1+x^r)(1+x^{r+1})}.$$

When $y > 0$, $x = e^{-2y} \in (0, 1)$. Therefore,

$$\begin{aligned} \sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) &= 4e^{-y} \frac{e^{-2y}}{1-e^{-4y}} \\ &= 2e^{-y} \frac{2}{e^{2y} - e^{-2y}} \\ &= 2e^{-y} \operatorname{cosech}(2y) \end{aligned}$$

as desired.

Notice that for all $x \in \mathbb{R}$, $\cosh x = \cosh(-x)$, therefore $\operatorname{sech} x = \operatorname{sech}(-x)$.

Therefore,

$$\begin{aligned}
& \sum_{r=-\infty}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) + \sum_{r=-\infty}^0 \operatorname{sech}(ry) \operatorname{sech}((r+1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) + \sum_{r=0}^{+\infty} \operatorname{sech}(-ry) \operatorname{sech}((-r+1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) + \sum_{r=0}^{+\infty} \operatorname{sech}(ry) \operatorname{sech}((r-1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) + \sum_{r=2}^{+\infty} \operatorname{sech}(ry) \operatorname{sech}((r-1)y) + \operatorname{sech}(y) \operatorname{sech}(0) + \operatorname{sech}(0) \operatorname{sech}(-y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) + \sum_{r=1}^{+\infty} \operatorname{sech}((r+1)y) \operatorname{sech}(ry) + 2 \operatorname{sech} y \\
&= 4e^{-y} \operatorname{cosech}(2y) + 2 \operatorname{sech} y \\
&= \frac{4e^{-y}}{\sinh 2y} + \frac{2}{\cosh y} \\
&= \frac{2e^{-y}}{\sinh y \cosh y} + \frac{2}{\cosh y} \\
&= \frac{2e^{-y} + 2 \sinh y}{\sinh y \cosh y} \\
&= \frac{2e^{-y} + e^y - e^{-y}}{\sinh y \cosh y} \\
&= \frac{e^y - e^{-y}}{\sinh y \cosh y} \\
&= \frac{2 \cosh y}{\sinh y \cosh y} \\
&= 2 \operatorname{cosech} y.
\end{aligned}$$

2016.3 Question 5

1. By the binomial theorem, we have

$$(1+x)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} x^k.$$

If we let $x = 1$, we have

$$2^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k}.$$

Since $\binom{2m+1}{m}$ is a part of the sum, and all the other terms are positive, and there are other terms which are not $\binom{2m+1}{m}$ (e.g. $\binom{2m+1}{0} = 1$), we therefore must have

$$\binom{2m+1}{m} < 2^{2m+1}.$$

2. Notice that

$$\begin{aligned} \binom{2m+1}{m} &= \frac{(2m+1)!}{m!(m+1)!} \\ &= \frac{(2m+1)(2m)(2m-1)\cdots(m+2)}{m!} \end{aligned}$$

A number theory argument follows. First, notice that all terms in the product $P_{m+1,2m+1}$ are within the numerator. Therefore, we must have

$$P_{m+1,2m+1} \mid (2m+1)(2m)(2m-1)\cdots(m+2).$$

Next, since all the terms in the product are primes, none of the terms will therefore have factors between 1 and m . This means that

$$\gcd(P_{m+1,2m+1}, m!) = 1,$$

i.e. $P_{m+1,2m+1}$ are coprime.

Therefore, given that $\binom{2m+1}{m} = \frac{(2m+1)(2m)(2m-1)\cdots(m+2)}{m!}$ is an integer, we must therefore have

$$P_{m+1,2m+1} \mid \binom{2m+1}{m},$$

and hence

$$P_{m+1,2m+1} \leq \binom{2m+1}{m} < 2^{2m},$$

as desired.

3. Notice that

$$\begin{aligned} P_{1,2m+1} &= P_{1,m+1} \cdot P_{m+1,2m+1} \\ &< 4^{m+1} \cdot 2^{2m} \\ &= 4^{m+1} \cdot 4^m \\ &= 4^{2m+1}, \end{aligned}$$

as desired.

4. First we look at the base case when $n = 2$.

$P_{1,2} = 2$, $4^2 = 16$, the original statement holds when $n = 2$.

Now, we use strong induction. Suppose the statement holds up to some $n = k \geq 2$.

If $k = 2m$ is even, the induction step for $2m \rightarrow 2m + 1$ is already shown in the previous part.

If $k = 2m + 1$ is odd, we must have that $k + 1$ is even. The only even prime is 2, but since $k \geq 2$, $k + 1 \neq 2$, and $k + 1$ must be composite.

Therefore, $P_{1,k+1} = P_{1,k} < 4^k < 4^{k+1}$. This completes the induction step.

Therefore, by strong induction, the statement $P_{1,n} < 4^n$ holds for all $n \geq 2$.

2016.3 Question 6

- In the case where $B > A > 0$ or $-B < -A < 0$, notice that

$$R \cosh(x + \gamma) = R \cosh x \cosh \gamma + R \sinh x \sinh \gamma.$$

Therefore, we would like $R \sinh \gamma = A$ and $R \cosh \gamma = B$.

Since $\cosh^2 \gamma - \sinh^2 \gamma = 1$, we have $R^2 = B^2 - A^2$.

We also have $\tanh \gamma = \frac{A}{B}$, and therefore $\gamma = \operatorname{artanh} \frac{A}{B}$.

Notice that $\cosh \gamma > 0$, so R must have the same sign as B .

- If $B > A > 0$, $R = \sqrt{B^2 - A^2}$.
- If $B < -A < 0$, $R = -\sqrt{B^2 - A^2}$.

- In the case where $-A < B < A$, notice that

$$R \sinh(x + \gamma) = R \sinh \gamma \cosh x + R \cosh \gamma \sinh x.$$

Therefore, we would like $R \cosh \gamma = A$ and $R \sinh \gamma = B$.

Since $\cosh^2 \gamma - \sinh^2 \gamma = 1$, we have $R^2 = B^2 - A^2$.

We also have $\tanh \gamma = \frac{B}{A}$, and therefore $\gamma = \operatorname{artanh} \frac{B}{A}$.

Notice that $\cosh \gamma > 0$, so R will have the same sign as A , and hence $R = \sqrt{A^2 - B^2}$.

- When $B = A$, we have

$$\begin{aligned} A \sinh x + B \cosh x &= A \frac{e^x - e^{-x}}{2} + A \frac{e^x + e^{-x}}{2} \\ &= Ae^x. \end{aligned}$$

- When $B = -A$, we have

$$\begin{aligned} A \sinh x + B \cosh x &= A \frac{e^x - e^{-x}}{2} - A \frac{e^x + e^{-x}}{2} \\ &= Ae^{-x}. \end{aligned}$$

Therefore, in conclusion,

$$A \sinh x + B \cosh x = \begin{cases} \sqrt{B^2 - A^2} \cosh \left(x + \operatorname{artanh} \frac{A}{B} \right), & 0 < A < B, \\ Ae^x, & 0 < B = A, \\ \sqrt{A^2 - B^2} \sinh \left(x + \operatorname{artanh} \frac{B}{A} \right), & -A < B < A, \\ -Ae^{-x}, & B = -A < 0, \\ -\sqrt{B^2 - A^2} \cosh \left(x + \operatorname{artanh} \frac{A}{B} \right), & -B < -A < 0. \end{cases}$$

1. We have $\operatorname{sech} x = a \tanh x + b$, and hence $1 = a \sinh x + b \cosh x$. If $b > a > 0$, we have

$$\sqrt{b^2 - a^2} \cosh \left(x + \operatorname{artanh} \frac{a}{b} \right) = 1.$$

Therefore,

$$\begin{aligned} \cosh \left(x + \operatorname{artanh} \frac{a}{b} \right) &= \frac{1}{\sqrt{b^2 - a^2}} \\ x + \operatorname{artanh} \frac{a}{b} &= \pm \operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} \\ x &= \pm \operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} - \operatorname{artanh} \frac{a}{b}, \end{aligned}$$

as desired.

2. When $a > b > 0$,

$$\sqrt{a^2 - b^2} \sinh \left(x + \operatorname{artanh} \frac{b}{a} \right) = 1.$$

Therefore,

$$\begin{aligned} \sinh \left(x + \operatorname{artanh} \frac{b}{a} \right) &= \frac{1}{\sqrt{a^2 - b^2}} \\ x + \operatorname{artanh} \frac{b}{a} &= \operatorname{arsinh} \frac{1}{\sqrt{a^2 - b^2}} \\ x &= \operatorname{arsinh} \frac{1}{\sqrt{a^2 - b^2}} - \operatorname{artanh} \frac{b}{a}. \end{aligned}$$

3. We would like to have two solutions to the equation $1 = a \sinh x + b \cosh x$.

- $0 < a < b$, this gives

$$x = \pm \operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} - \operatorname{artanh} \frac{a}{b},$$

For this to make sense, we must have $\frac{1}{\sqrt{b^2 - a^2}} \geq 1$, and therefore $0 < \sqrt{b^2 - a^2} \leq 1$, which is $0 < b^2 - a^2 \leq 1$.

For this to have two distinct points, we would like to have $\operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} \neq 0$ as well. This means $b^2 - a^2 \neq 1$.

Therefore, in this case, this means that $a < b < \sqrt{a^2 + 1}$.

- $b = a$, this gives $ae^x = 1$, which gives a unique solution $x = -\ln a$.
- $-a < b < a$, this gives

$$\sqrt{A^2 - B^2} \sinh \left(x + \operatorname{artanh} \frac{B}{A} \right) = 1,$$

which can only give the solution $x = \operatorname{arsinh} \frac{1}{\sqrt{A^2 - B^2}} - \operatorname{artanh} \frac{B}{A}$.

- $b = -a$, this gives $-ae^{-x} = 1$, which does not have a solution.
- $-b < -a < 0$, this gives

$$-\sqrt{b^2 - a^2} \cosh \left(x + \operatorname{artanh} \frac{a}{b} \right) = 1,$$

but this is impossible, since both square root and cosh are always positive.

Therefore, the only possibility is when $a < b < \sqrt{a^2 + 1}$.

4. When they touch at a point, this will mean at this value, the number of solutions will change on both sides. This is only possible when $b = \sqrt{a^2 + 1}$.

Therefore,

$$x = -\operatorname{artanh} \frac{a}{\sqrt{a^2 + 1}}.$$

Hence,

$$\begin{aligned} y &= a \tanh x + b \\ &= -a \cdot \frac{a}{\sqrt{a^2 + 1}} + \sqrt{a^2 + 1} \\ &= \frac{-a^2 + a^2 + 1}{\sqrt{a^2 + 1}} \\ &= \frac{1}{\sqrt{a^2 + 1}}. \end{aligned}$$

2016.3 Question 7

For $\omega = \exp \frac{2\pi i}{n}$, we have for $k = 0, 1, 2, \dots, n-1$, that $\omega^k = \exp \frac{2\pi i k}{n}$. Therefore,

$$(\omega^k)^n = \exp \frac{2\pi i k n}{n} = \exp(2\pi i k) = 1.$$

Also, notice that $\arg \omega^k = \frac{2k\pi}{n}$, which means that all ω^k 's are different.

This means that $\omega^0 = 1, \omega^1 = \omega, \omega^2, \dots, \omega^{n-1}$ are exactly the n roots to the polynomial $z^n - 1$, which has leading coefficient 1.

Therefore, we must have

$$(z-1)(z-\omega)\cdots(z-\omega^{n-1}) = z^n - 1,$$

as desired.

For the following parts, W.L.O.G. let the orientation of the polygon be such that $X_k = \omega^k$.

1. Let z represent the complex number for P , we have

$$\begin{aligned} \prod_{k=0}^{n-1} |PX_k| &= \prod_{k=0}^{n-1} |z - \omega^k| \\ &= \left| \prod_{k=0}^{n-1} (z - \omega^k) \right| \\ &= |z^n - 1|. \end{aligned}$$

Since P is equidistant from X_0 and X_1 , we must have that $P = r \exp\left(\frac{\pi i}{n}\right)$ for some $r \in \mathbb{R}$, where $|r| = |OP|$. Therefore, we have

$$\begin{aligned} \prod_{k=0}^{n-1} |PX_k| &= |z^n - 1| \\ &= \left| r^n \exp\left(\frac{\pi i}{n}\right) - 1 \right| \\ &= |-r^n - 1| \\ &= |r^n + 1|. \end{aligned}$$

If n is even, then $r^n = |r|^n > 0$, and therefore $|r^n + 1| = r^n + 1 = |r|^n + 1 = |OP|^n + 1$ as desired.

If n is odd, and $r > 0$, then $r^n = |r|^n > 0$, and

$$\begin{aligned} \text{LHS} &= |r^n + 1| \\ &= r^n + 1 \\ &= |r|^n + 1 \\ &= |OP|^n + 1. \end{aligned}$$

When $-1 \leq r < 0$, we have $-1 \leq r^n = -|r|^n < 0$, and

$$\begin{aligned} \text{LHS} &= |r^n + 1| \\ &= r^n + 1 \\ &= -|r|^n + 1 \\ &= -|OP|^n + 1. \end{aligned}$$

When $r < -1$, we have $r^n = -|r|^n < -1$, and

$$\begin{aligned} \text{LHS} &= |r^n + 1| \\ &= -r^n - 1 \\ &= |r|^n - 1 \\ &= |OP|^n - 1. \end{aligned}$$

In summary, when n is odd, we have

$$\prod_{k=0}^{n-1} |PX_k| = \begin{cases} |OP|^n + 1, & P \text{ is in the first quadrant,} \\ -|OP|^n + 1, & P \text{ is in the third quadrant and } |OP| \leq 1, \\ |OP|^n - 1, & P \text{ is in the third quadrant and } |OP| > 1. \end{cases}$$

2. Notice that for a general point P whose complex number is z , we have

$$\begin{aligned} \prod_{k=1}^{n-1} |PX_k| &= (z - \omega)(z - \omega^2) \cdots (z - \omega^{n-1}) \\ &= \frac{z^n - 1}{z - 1} \\ &= 1 + z + z^2 + \cdots + z^{n-1}. \end{aligned}$$

If we let $P = X_0$, $z = 1$, and $\text{RHS} = n$, just as we desired.

2016.3 Question 8

1. If we replace x with $-x$ in the original equation, we get

$$f(-x) + (1 - (-x))f(-(-x)) = (-x)^2,$$

which simplifies to

$$f(-x) + (1 + x)f(x) = x^2$$

as desired.

Therefore, we have a pair of equations in terms of $f(x)$ and $f(-x)$:

$$\begin{cases} f(x) + (1 - x)f(-x) = x^2 \\ (1 + x)f(x) + f(-x) = x^2. \end{cases}$$

Multiplying the second equation by $(1 - x)$ gives us

$$(1 - x^2)f(x) + (1 - x)f(-x) = x^2(1 - x),$$

and subtracting the first equation from this

$$-x^2f(x) = -x^3,$$

which gives $f(x) = x$.

Plugging this back, we have

$$\begin{aligned} \text{LHS} &= f(x) + (1 - x)f(-x) \\ &= x + (1 - x)(-x) \\ &= x - x + x^2 \\ &= x^2 \\ &= \text{RHS} \end{aligned}$$

which holds. Therefore, $f(x) = x$ is the solution to the functional equation.

2. For $x \neq 1$, we have

$$\begin{aligned} K(K(x)) &= \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} \\ &= \frac{(x+1) + (x-1)}{(x+1) - (x-1)} \\ &= \frac{2x}{2} \\ &= x, \end{aligned}$$

for $x \neq 1$, as desired.

The equation on g is

$$g(x) + xg(K(x)) = x,$$

and if we substitute x as $K(x)$, we have

$$g(K(x)) + K(x)g(K(K(x))) = K(x),$$

which simplifies to

$$g(K(x)) + K(x)g(x) = K(x).$$

Multiplying the second equation by x , we have

$$xK(x)g(X) + xg(K(x)) = xK(x),$$

and subtracting the first equation from this gives

$$(xK(x) - 1)g(x) = x(K(x) - 1),$$

which gives

$$\begin{aligned} g(x) &= \frac{x(K(x) - 1)}{xK(x) - 1} \\ &= \frac{x\left(\frac{x+1}{x-1} - 1\right)}{x \cdot \frac{x+1}{x-1} - 1} \\ &= \frac{x[(x+1) - (x-1)]}{x(x+1) - (x-1)} \\ &= \frac{2x}{x^2 + 1}, \end{aligned}$$

for $x \neq 1$.

If we plug this back to the original equation, we have

$$\begin{aligned} \text{LHS} &= \frac{2x}{x^2 + 1} + x \frac{2 \cdot \frac{x+1}{x-1}}{\left(\frac{x+1}{x-1}\right)^2 + 1} \\ &= \frac{2x}{x^2 + 1} + \frac{2x \cdot (x+1) \cdot (x-1)}{(x+1)^2 + (x-1)^2} \\ &= \frac{2x}{x^2 + 1} + \frac{2x(x^2 - 1)}{2x^2 + 2} \\ &= \frac{2x}{x^2 + 1} + \frac{x(x^2 - 1)}{x^2 + 1} \\ &= \frac{x^3 - x + 2x}{x^2 + 1} \\ &= \frac{x(x^2 + 1)}{x^2 + 1} \\ &= x \\ &= \text{RHS}, \end{aligned}$$

so

$$g(x) = \frac{2x}{x^2 + 1}$$

is the solution to the original functional equation.

3. Let $H(x) = \frac{1}{1-x}$. Notice that

$$\begin{aligned} H(H(x)) &= \frac{1}{1 - \frac{1}{1-x}} \\ &= \frac{1-x}{1-x-1} \\ &= \frac{x-1}{x} \\ &= 1 - \frac{1}{x} \end{aligned}$$

and

$$\begin{aligned} H(H(H(x))) &= \frac{1}{1 - \left(1 - \frac{1}{x}\right)} \\ &= \frac{x}{1} \\ &= x. \end{aligned}$$

Now, if we replace all the x with $\frac{1}{1-x}$, we will get

$$h\left(\frac{1}{1-x}\right) + h\left(1 - \frac{1}{x}\right) = 1 - \frac{1}{1-x} - \left(1 - \frac{1}{x}\right),$$

and doing the same replacement again gives us

$$h\left(1 - \frac{1}{x}\right) + h(x) = 1 - \left(1 - \frac{1}{x}\right) - x.$$

Summing these two equations, together with the original equation, gives us that

$$2 \cdot \left[h\left(\frac{1}{1-x}\right) + h\left(1 - \frac{1}{x}\right) + h(x) \right] = 3 - 2 \cdot \left[x + \frac{1}{1-x} + \left(1 - \frac{1}{x}\right) \right],$$

and therefore

$$h\left(\frac{1}{1-x}\right) + h\left(1 - \frac{1}{x}\right) + h(x) = \frac{3}{2} - \left[x + \frac{1}{1-x} + \left(1 - \frac{1}{x}\right) \right].$$

Subtracting the second equation from this, gives that

$$\begin{aligned} h(x) &= \left(\frac{3}{2} - \left[x + \frac{1}{1-x} + \left(1 - \frac{1}{x}\right) \right] \right) - \left[1 - \frac{1}{1-x} - \left(1 - \frac{1}{x}\right) \right] \\ &= \frac{1}{2} - x. \end{aligned}$$

Plugging this back to the original equation, we have

$$\begin{aligned} \text{LHS} &= \frac{1}{2} - x + \frac{1}{2} - \frac{1}{1-x} \\ &= 1 - x - \frac{1}{1-x} \\ &= \text{RHS}, \end{aligned}$$

which satisfies the original functional equation. Therefore, the original equation solves to

$$h(x) = \frac{1}{2} - x.$$

2016.3 Question 12

1. Let $X \sim B(100n, 0.2)$. We have $\mu = 100n \cdot 0.2 = 20n$, and $\sigma^2 = 100n \cdot 0.2 \cdot 0.8 = 16n$.

We have that

$$\begin{aligned}
 \alpha &= P(16n \leq X \leq 24n) \\
 &= P(|X - 20n| \leq 4n) \\
 &= P(|X - \mu| \leq \sigma\sqrt{n}) \\
 &= 1 - P(|X - \mu| > \sigma\sqrt{n}) \\
 &\geq 1 - \frac{1}{\sqrt{n^2}} \\
 &= 1 - \frac{1}{n},
 \end{aligned}$$

as desired, where we applied the Chebyshev Inequality for $k = \sqrt{n} > 0$.

2. Let $X \sim \text{Po}(n)$. Therefore, $\mu = E(X) = n$, $\sigma = \sqrt{\text{Var}(X)} = \sqrt{n}$. To show the desired inequality is equivalent to showing that

$$\frac{1 + n + \frac{n^2}{2!} + \dots + \frac{n^{2n}}{(2n)!}}{e^n} \geq 1 - \frac{1}{n}.$$

Notice that the left-hand side is simply $P(0 \leq X \leq 2n)$. By the Chebyshev Inequality, we have

$$\begin{aligned}
 \text{LHS} &= P(0 \leq X \leq 2n) \\
 &= P(|X - \mu| \leq n) \\
 &= P(|X - \mu| \leq \sqrt{n}\sigma) \\
 &= 1 - P(|X - \mu| > \sqrt{n}\sigma) \\
 &\geq 1 - \frac{1}{n} \\
 &= \text{RHS},
 \end{aligned}$$

as desired, where we applied the Chebyshev Inequality for $k = \sqrt{n} > 0$.

2016.3 Question 13

For a random variable X with $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$, we have

$$\kappa(X) = \frac{E[(X - \mu)^4]}{\sigma^4} - 3$$

We have $Y = X - a$. Therefore, $E(Y) = \mu - a$ and $\text{Var}(Y) = \sigma^2$.

$$\begin{aligned} \kappa(Y) &= \frac{E[(Y - (\mu - a))^4]}{\sigma^4} - 3 \\ &= \frac{E[((X - a) - (\mu - a))^4]}{\sigma^4} - 3 \\ &= \frac{E[(X - \mu)^4]}{\sigma^4} - 3 \\ &= \kappa(X), \end{aligned}$$

as desired.

1. Let $X \sim N(0, \sigma^2)$, $\mu = 0$. Notice that

$$\kappa(X) = \frac{E(X^4)}{\sigma^4} - 3.$$

X has p.d.f.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Therefore,

$$E(X^4) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right) dx.$$

Now, consider using integration by parts. Notice that

$$d \exp\left(-\frac{x^2}{2\sigma^2}\right) = -\frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx,$$

and therefore, using integration by parts, we have

$$\begin{aligned} &\int x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\ &= -\sigma^2 \int x^3 d \exp\left(-\frac{x^2}{2\sigma^2}\right) \\ &= -\sigma^2 \left[x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right) - \int \exp\left(-\frac{x^2}{2\sigma^2}\right) d(x^3) \right] \\ &= 3\sigma^2 \int x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right) dx - \sigma^2 x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right). \end{aligned}$$

Therefore, considering the definite integral, we have

$$\begin{aligned} E(X^4) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\ &= \frac{\sigma}{\sqrt{2\pi}} \left[3 \int_{-\infty}^{+\infty} x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right) dx - \left[x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right) \right]_{-\infty}^{+\infty} \right] \\ &= \frac{\sigma}{\sqrt{2\pi}} \left[3 \cdot \sigma\sqrt{2\pi} \cdot \sigma^2 - 0 \right] \\ &= 3\sigma^4. \end{aligned}$$

Therefore,

$$\kappa(X) = \frac{E(X^4)}{\sigma^4} - 3 = \frac{3\sigma^4}{\sigma^4} - 3 = 0,$$

as desired.

An alternative solution exists using generating functions.

Recall that a general normal distribution $N(\mu, \sigma^2)$ has MGF

$$M(t) = \exp\left(\mu t + \frac{\sigma^2}{2}t^2\right),$$

and hence

$$\begin{aligned} M_X(t) &= \exp\left(\frac{\sigma^2}{2}t^2\right) \\ &= 1 + \left(\frac{\sigma^2}{2}t^2\right) + \frac{\left(\frac{\sigma^2}{2}t^2\right)^2}{2!} + \dots \end{aligned}$$

Therefore,

$$E(X^4) = M_X^{(4)}(0) = \left(\frac{\sigma^2}{2}\right)^4 \cdot 4! = 3\sigma^4,$$

and the result follows.

2. Notice that

$$\begin{aligned} T^4 &= \sum_a \binom{4}{4} Y_a^4 + \sum_{a < b} \left[\binom{4}{1, 3} Y_a Y_b^3 + \binom{4}{2, 2} Y_a^2 Y_b^2 \right] \\ &\quad + \sum_{a < b < c} \binom{4}{1, 1, 2} Y_a Y_b Y_c^2 + \sum_{a < b < c < d} \binom{4}{1, 1, 1, 1} Y_a Y_b Y_c Y_d \\ &= \sum_a Y_a^4 + \sum_{a < b} (4Y_a Y_b^3 + 6Y_a^2 Y_b^2) + \sum_{a < b < c} 12Y_a Y_b Y_c^2 + \sum_{a < b < c < d} 24Y_a Y_b Y_c Y_d, \end{aligned}$$

where

$$\binom{n}{a_1, a_2, \dots, a_k} = \frac{n!}{a_1! a_2! \dots a_k!}, \quad \sum_{i=1}^k a_i = n$$

stands for the multinomial coefficient.

Note that $E(Y_r) = 0$ for any $r = 1, 2, \dots, n$. Therefore,

$$\begin{aligned} E(Y_a Y_b^3) &= E(Y_a) E(Y_b^3) = 0, \\ E(Y_a Y_b Y_c^2) &= E(Y_a) E(Y_b) E(Y_c^2) = 0, \\ E(Y_a Y_b Y_c Y_d) &= E(Y_a) E(Y_b) E(Y_c) E(Y_d) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} E(T^4) &= \sum_a E(Y_a^4) + \sum_{a < b} 6 E(Y_a^2 Y_b^2) \\ &= \sum_{r=1}^n E(Y_r^4) + 6 \sum_{r=1}^{n-1} \sum_{s=r+1}^n E(Y_r^2) E(Y_s^2), \end{aligned}$$

as desired.

3. Let $Y_i = X_i - \mu$ for $i = 1, 2, \dots, n$, and $\mu = E(X)$, $\sigma^2 = \text{Var}(X) = \text{Var}(Y)$ with $E(Y) = 0$

Therefore, let $T = \sum_i^n Y_i = \sum_i^n X_i - n\mu$, we must have $E(T) = 0$ and $\text{Var}(T) = n\sigma^2$.

But since the kurtosis remains constant with shifts, we must have that $\kappa(Y_i) = \kappa$, and

$$\kappa(T) = \kappa \left[\sum_i^n X_i \right].$$

Hence, we have

$$\begin{aligned} \kappa \left[\sum_i^n X_i \right] &= \kappa(T) \\ &= \frac{\mathbb{E}(T^4)}{(n\sigma^2)^2} - 3 \\ &= \frac{\sum_{r=1}^n \mathbb{E}(Y_r^4) + 6 \sum_{r=1}^{n-1} \sum_{s=r+1}^n \mathbb{E}(Y_a^2) \mathbb{E}(Y_b^2)}{n^2 \sigma^4} - 3 \\ &= \frac{1}{n^2} \sum_{r=1}^n \frac{\mathbb{E}(Y_r^4)}{\sigma^4} + \frac{6}{n^2} \sum_{r=1}^{n-1} \sum_{s=r+1}^n \frac{\sigma^4}{\sigma^4} - 3 \\ &= \frac{1}{n^2} n \cdot (\kappa + 3) + \frac{6}{n^2} \binom{n}{2} - 3 \\ &= \frac{\kappa}{n} + \frac{3n + 3n(n-1) - 3n^2}{n^2} \\ &= \frac{\kappa}{n} + 0 \\ &= \frac{\kappa}{n}, \end{aligned}$$

as desired.