2013 Paper 3

2013.3 Question 1

Since $t = \tan \frac{1}{2}x$, we have

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\frac{1}{2}x = \frac{1}{2}(1+\tan^2\frac{1}{2}x) = \frac{1}{2}(1+t^2).$$

By the tangent double-angle formula, we have

$$\tan x = \frac{2t}{1-t^2},$$

and hence

$$\cot x = \frac{1-t^2}{2t}.$$

Therefore,

$$\csc^2 x = 1 + \cot^2 x = 1 + \frac{(1-t^2)^2}{(2t)^2} = \frac{(1+t^2)^2}{(2t)^2},$$

which means

$$\sin^2 x = \frac{(2t)^2}{(1+t^2)^2},$$

and hence

$$|\sin x| = \frac{2t}{1+t^2}.$$

What remains is to consider the sign. Notice that $t \geq 0$ if and only if

$$\frac{x}{2} \in \bigcup_{k\in\mathbb{Z}} \left[k\pi, k\pi + \frac{\pi}{2}\right),$$

which is

$$x \in \bigcup_{k\in\mathbb{Z}} [2k\pi, 2k\pi+\pi),$$

but this is also precisely if and only if $\sin x \geq 0$.

This means $\sin x$ must take the same sign as t, and hence

$$\sin x = \frac{2t}{1+t^2}.$$

Using this substitution, we have when $x = 0, t = 0$ and when $x = \frac{1}{2}\pi, t = 1$, and also

$$\mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}.$$

This means

$$\begin{aligned}
I &= \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}x}{1+a\sin x} \\
&= \int_0^1 \frac{\frac{2\,\mathrm{d}t}{1+t^2}}{1+a\cdot\frac{2t}{1+t^2}} \\
&= \int_0^1 \frac{2\,\mathrm{d}t}{1+2at+t^2} \\
&= \int_0^1 \frac{2\,\mathrm{d}t}{(t+a)^2+(1-a^2)} \\
&= \frac{2}{1-a^2}\int_0^1 \frac{\mathrm{d}t}{\left(\frac{t+a}{\sqrt{1-a^2}}\right)^2+1} \\
&= \frac{2}{1-a^2}\cdot\sqrt{1-a^2}\cdot\left[\arctan\left(\frac{t+a}{\sqrt{1-a^2}}\right)\right]_0^1 \\
&= \frac{2}{\sqrt{1-a^2}}\cdot\left[\arctan\left(\frac{1+a}{\sqrt{1-a^2}}\right) - \arctan\left(\frac{a}{\sqrt{1-a^2}}\right)\right].
\end{aligned}$$

But notice that

$$\begin{aligned}\arctan\left(\frac{1+a}{\sqrt{1-a^2}}\right)-\arctan\left(\frac{a}{\sqrt{1-a^2}}\right)&=\arctan\left(\frac{\frac{1+a}{\sqrt{1-a^2}}-\frac{a}{\sqrt{1-a^2}}}{1+\frac{1+a}{\sqrt{1-a^2}}\cdot\frac{a}{\sqrt{1-a^2}}}\right)\\&=\arctan\left(\frac{\frac{1}{\sqrt{1-a^2}}}{1+\frac{a+a^2}{1-a^2}}\right)\\&=\arctan\left(\frac{\sqrt{1-a^2}}{(1-a^2)+(a+a^2)}\right)\\&=\arctan\left(\frac{\sqrt{1-a}\cdot\sqrt{1+a}}{1+a}\right)\\&=\arctan\left(\frac{\sqrt{1-a}}{\sqrt{1+a}}\right),\end{aligned}$$

and hence

$$I=\frac{2}{\sqrt{1-a^2}}\arctan\left(\frac{\sqrt{1-a}}{\sqrt{1+a}}\right),$$

as desired.

We have

$$\begin{aligned}I_{n+1}+2I_n&=\int_0^{\frac{1}{2}\pi}\frac{\sin^{n+1}x+2\sin^n x}{2+\sin x}\,\mathrm{d}x\\&=\int_0^{\frac{1}{2}\pi}\sin^n x\,\mathrm{d}x.\end{aligned}$$

Therefore, we have

$$\begin{aligned}I_3+2I_2&=\int_0^{\frac{1}{2}\pi}\sin^2 x\,\mathrm{d}x\\&=\int_0^{\frac{1}{2}\pi}\frac{1-\cos 2x}{2}\,\mathrm{d}x\\&=\left[\frac{1}{2}\cdot x-\frac{1}{4}\sin 2x\right]_0^{\frac{1}{2}\pi}\\&=\left(\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{4}\sin\pi\right)-\left(\frac{1}{4}\sin 0-\frac{1}{2}\cdot 0\right)\\&=\frac{\pi}{4},\end{aligned}$$

$$\begin{aligned}I_2+2I_1&=\int_0^{\frac{1}{2}\pi}\sin x\,\mathrm{d}x\\&=[-\cos x]_0^{\frac{1}{2}\pi}\\&=\left(-\cos\frac{1}{2}\pi\right)-(-\cos 0)\\&=(0)-(-1)\\&=1,\end{aligned}$$

and

$$\begin{aligned}I_1+2I_0&=\int_0^{\frac{1}{2}\pi}\sin^0 x\,\mathrm{d}x\\&=[x]_0^{\frac{1}{2}\pi}\\&=\frac{1}{2}\pi.\end{aligned}$$

Also, notice that

$$\begin{aligned}
I_0 &= \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}x}{2+\sin x} \\
&= \frac{1}{2}\int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}x}{1+\frac{1}{2}\sin x} \\
&= \frac{1}{2} \cdot \frac{2}{\sqrt{1-\left(\frac{1}{2}\right)^2}} \cdot \arctan \frac{\sqrt{1-\frac{1}{2}}}{\sqrt{1+\frac{1}{2}}} \\
&= \frac{1}{2} \cdot \frac{4}{\sqrt{3}} \cdot \arctan \frac{1}{\sqrt{3}} \\
&= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} \\
&= \frac{\pi}{3\sqrt{3}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_3 &= \frac{\pi}{4} - 2I_2 \\
&= \frac{\pi}{4} - 2 \cdot (1 - 2I_1) \\
&= \frac{\pi}{4} - 2 + 4I_1 \\
&= \frac{\pi}{4} - 2 + 4\left(\frac{1}{2}\pi - 2I_0\right) \\
&= \frac{\pi}{4} - 2 + 2\pi - 8I_0 \\
&= \frac{9\pi}{4} - 2 - \frac{8\pi}{3\sqrt{3}} \\
&= \left(\frac{9}{4} - \frac{8}{3\sqrt{3}}\right)\pi - 2.
\end{aligned}$$

2013.3 Question 2

We must have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x} \cdot \frac{\arcsin x}{\sqrt{1-x^2}} \\ &= \frac{1}{1-x^2} \cdot \left(\frac{1}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} - \arcsin x \cdot (-2x) \cdot \left(\frac{1}{2}\right) \cdot \frac{1}{\sqrt{1-x^2}}\right) \\ &= \frac{1}{1-x^2} \cdot \left(1 + x \cdot \frac{\arcsin x}{\sqrt{1-x^2}}\right) \\ &= \frac{1}{1-x^2} \cdot (1+xy),\end{aligned}$$

which gives

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy - 1 = (1+xy) - xy - 1 = 0$$

as desired.

Differentiating both sides of this equation w.r.t. x gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \cdot (1-x^2) - 2x \cdot \frac{\mathrm{d}y}{\mathrm{d}x} - y - x\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

which combined gives

$$(1-x^2) \cdot \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3x \cdot \frac{\mathrm{d}y}{\mathrm{d}x} - y = 0.$$

If we extend the definition of the differentiation operator to

$$\frac{\mathrm{d}^0y}{\mathrm{d}x^0} = y,$$

then this precisely proves the desired statement for the case $n = 0$ since $2n + 3 = 3$ and $(n+1)^2 = 1$, and we will prove the desired statement for all non-negative integer n. The base case is shown as above.

Now, assume the given holds for some $n = k$ where k is a non-negative integer, i.e.

$$(1-x^2) \cdot \frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} - (2k+3)x \cdot \frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} - (k+1)^2 \cdot \frac{\mathrm{d}^ky}{\mathrm{d}x^k} = 0,$$

we aim to show that the same holds for $n = k + 1$.

Differentiating both sides with respect to x gives

$$(-2x) \cdot \frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} + (1-x^2) \cdot \frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - (2k+3) \cdot \frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} - (2k+3)x \cdot \frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} - (k+1)^2 \cdot \frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = 0,$$

which then simplifies to

$$(1-x^2) \cdot \frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - (2k+5)x \cdot \frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} - (k^2+4k+4) \cdot \frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = 0.$$

But notice that $n+2 = (k+1)+2 = k+3$, $n+1 = (k+1)+1 = k+2$, $(n+1)^2 = (k+2)^2 = k^2+4k+4$, $2n + 3 = 2(k + 1) + 3 = 2k + 5$, so this is exactly the statement when $n = k + 1$, which finishes our inductive step.

Hence, by the Principle of Mathematical Induction, we can conclude that the original statement holds for any non-negative integer n, and hence for any positive integer n.

We have that

$$y|_{x=0} = \frac{\arcsin 0}{\sqrt{1-0^2}} = \frac{0}{1} = 0,$$

and evaluating the equation on the first derivative at $x = 0$ gives

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 1.$$

Evaluating the proven equation at $x = 0$ gives

$$\left.\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}\right|_{x=0} = (n+1)^2 \left.\frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right|_{x=0}.$$

Using this, we can conclude that

$$\left.\frac{\mathrm{d}^{2r}y}{\mathrm{d}x^{2r}}\right|_{x=0} = 0$$

for all $r \geq 0$ where r is an integer, since it is 0 when $n = 0$, and that

$$\left.\frac{\mathrm{d}^{2r+1}y}{\mathrm{d}x^{2r+1}}\right|_{x=0} = ((2r)!!)^2 = 2^{2r} \cdot (r!)^2$$

for all $r \geq 0$ where r is an integer, by mathematical induction.

Hence, the MacLaurin Series for $\frac{\arcsin x}{\sqrt{1-x^2}}$, must be

$$\begin{aligned}
\frac{\arcsin x}{\sqrt{1-x^2}} &= \sum_{k=0}^{\infty} \frac{\left.\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right|_{x=0}}{k!} \cdot x^k \\
&= \sum_{r=0}^{\infty} \frac{\left.\frac{\mathrm{d}^{2r} y}{\mathrm{d}x^{2r}}\right|_{x=0}}{(2r)!} \cdot x^{2r} + \sum_{r=0}^{\infty} \frac{\left.\frac{\mathrm{d}^{2r+1} y}{\mathrm{d}x^{2r+1}}\right|_{x=0}}{(2r+1)!} \cdot x^{2r+1} \\
&= 0 + \sum_{r=0}^{\infty} \frac{2^{2r} \cdot (r!)^2}{(2r+1)!} \cdot x^{2r+1} \\
&= \sum_{r=0}^{\infty} \frac{2^{2r} \cdot (r!)^2}{(2r+1)!} \cdot x^{2r+1}.
\end{aligned}$$

This means the general term for even powers of x is zero, and the general term for odd powers of x is

$$\frac{2^{2r} \cdot (r!)^2}{(2r+1)!} \cdot x^{2r+1}$$

where r is any non-negative integer.

The infinite sum can be expressed as

$$\sum_{r=0}^{\infty} \frac{(r!)^2}{(2r+1)!} = 2 \cdot \sum_{r=0}^{\infty} \frac{2^{2r} \cdot (r!)^2}{(2r+1)!} \cdot \left(\frac{1}{2}\right)^{2r+1},$$

which is precisely double the value of

$$\left[\frac{\arcsin x}{\sqrt{1-x^2}}\right]_{x=\frac{1}{2}} = \frac{\arcsin \frac{1}{2}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{\pi/6}{\sqrt{3}/2} = \frac{\pi}{3\sqrt{3}},$$

Hence, the sum evaluates to $\frac{2\pi}{3\sqrt{3}}$.

2013.3 Question 3

Since $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{4}_4 = \mathbf{0}$, we must have

$$\begin{aligned}0 &= \mathbf{0}\cdot\mathbf{0}\\ &= (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{4}_4)\cdot(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{4}_4)\\ &= \sum_{i=1}^{4}\mathbf{p}_i\cdot\mathbf{p}_i + 2\sum_{i=1}^{3}\sum_{j=i+1}^{4}\mathbf{p}_i\cdot\mathbf{p}_j.\end{aligned}$$

Since P_i are on the unit sphere, we must have $\mathbf{p}_i\cdot\mathbf{p}_i = 1$. By symmetry, for all $i \neq j$,

$$\mathbf{p}_i\cdot\mathbf{p}_j$$

must be some real constant, say k.

Hence,

$$0 = 4\cdot 1 + 2\cdot 6\cdot k,$$

which solves to

$$k = -\frac{1}{3},$$

as desired.

1. We have

$$\begin{aligned}\sum_{i=1}^{4}(XP_i)^2 &= \sum_{i=1}^{4}(\mathbf{p}_i - \mathbf{x})\cdot(\mathbf{p}_i - \mathbf{x})\\ &= \sum_{i=1}^{4}(\mathbf{p_i}\cdot\mathbf{p_i} - 2\mathbf{x}\cdot\mathbf{p}_i + \mathbf{x}\cdot\mathbf{x})\\ &= \sum_{i=1}^{4}\mathbf{p_i}\cdot\mathbf{p_i} - 2\mathbf{x}\cdot\sum_{i=1}^{4} + 4\cdot\mathbf{x}\cdot\mathbf{x}\\ &= \sum_{i=1}^{4}1 - 2\mathbf{x}\cdot\mathbf{0} + 4\cdot 1\\ &= 4 - 0 + 4\\ &= 8.\end{aligned}$$

2. Since $P_1(0,0,1)$ and $P_2(a,0,b)$, we must have

$$\mathbf{p}_1 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix}a\\0\\b\end{pmatrix},$$

and hence

$$\mathbf{p}_1\cdot\mathbf{p}_2 = 0\cdot a + 0\cdot 0 + 1\cdot b = b = -\frac{1}{3}.$$

We must have

$$|\mathbf{p}_2| = \sqrt{a^2 + 0^2 + b^2} = \sqrt{a^2 + b^2} = 1,$$

which means

$$a = \frac{2\sqrt{2}}{3},$$

as desired.

The z-component of $\mathbf{p}_3$ and $\mathbf{p}_4$ must also be $-\frac{1}{3}$, due to the dot product with $vectp_1$ being equal to the z-component must also be equal to $-\frac{1}{3}$.

Let

$$\mathbf{p}_3 = \begin{pmatrix}c\\d\\-\frac{1}{3}\end{pmatrix},$$

then from $\sum_{i=1}^{4} \mathbf{p}_i = \mathbf{0}$, we have

$$\mathbf{p}_4 = \begin{pmatrix} -c - \frac{2\sqrt{2}}{3} \\ -d \\ -\frac{1}{3} \end{pmatrix}.$$

Since $\mathbf{p}_3 \cdot \mathbf{p}_2 = -\frac{1}{3}$, we have

$$\frac{2\sqrt{2}}{3} \cdot c + 0 \cdot d + \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right) = -\frac{1}{3},$$

and hence

$$\frac{2\sqrt{2}}{3} c = -\frac{4}{9},$$

which means

$$6\sqrt{2}c = -4,$$

and hence

$$c = -\frac{4}{6\sqrt{2}} = -\frac{\sqrt{2}}{3}.$$

Now, since $\mathbf{p}_3 \cdot \mathbf{p}_4 = -\frac{1}{3}$, we have

$$c \cdot \left(-c - \frac{2\sqrt{2}}{3}\right) + d \cdot (-d) + \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right) = -\frac{1}{3}.$$

Therefore,

$$\left(-\frac{\sqrt{2}}{3}\right) \cdot \left(-\frac{\sqrt{2}}{3}\right) - d^2 = -\frac{4}{9},$$

and hence

$$d^2 = \frac{2}{3},$$

giving

$$d = \pm \frac{\sqrt{2}}{\sqrt{3}}.$$

Hence,

$$P_3 \left(-\frac{\sqrt{2}}{3}, \pm\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{3}\right), P_4 \left(-\frac{\sqrt{2}}{\sqrt{3}}, \mp\frac{\sqrt{2}}{3}, -\frac{1}{3}\right).$$

3. We have

$$\begin{aligned}
\sum_{i=1}^{4} (XP_i)^4 &= \sum_{i=1}^{4} \left[(\mathbf{p}_i - \mathbf{x}) \cdot (\mathbf{p}_i - \mathbf{x})\right]^2 \\
&= \sum_{i=1}^{4} \left(\mathbf{p}_i \cdot \mathbf{p}_i - 2\mathbf{x} \cdot \mathbf{p}_i + \mathbf{x} \cdot \mathbf{x}\right)^2 \\
&= \sum_{i=1}^{4} \left(1 + 1 - 2\mathbf{x} \cdot \mathbf{p}_i\right)^2 \\
&= \sum_{i=1}^{4} \left(2 - 2\mathbf{x} \cdot \mathbf{p}_i\right)^2 \\
&= 4 \sum_{i=1}^{4} \left(1 - \mathbf{x} \cdot \mathbf{p}_i\right)^2.
\end{aligned}$$

Let $X(x, y, z)$. We have

$$\begin{aligned}
\sum_{i=1}^{4} (XP_i)^4 &= 4 \sum_{i=1}^{4} (1 - \mathbf{x} \cdot \mathbf{p}_i)^2 \\
&= 4 \left[\left(1 - \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)^2 + \left(1 - \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} \frac{2\sqrt{2}}{3} \\ 0 \\ -\frac{1}{3} \end{pmatrix} \right)^2 \right. \\
&\quad \left. + \left(1 - \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{3} \end{pmatrix} \right)^2 + \left(1 - \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{3} \end{pmatrix} \right)^2 \right] \\
&= 4 \left[(1 - z)^2 + \left(1 - \frac{2\sqrt{2}}{3} x + \frac{1}{3} z \right)^2 \right. \\
&\quad \left. + \left(1 + \frac{\sqrt{2}}{3} x - \frac{\sqrt{2}}{\sqrt{3}} y + \frac{1}{3} z \right)^2 + \left(1 + \frac{\sqrt{2}}{3} x + \frac{\sqrt{2}}{\sqrt{3}} y + \frac{1}{3} z \right)^2 \right] \\
&= 4 \left(4 + \frac{4}{3} x^2 + \frac{4}{3} y^2 + \frac{4}{3} z^2 \right) \\
&= 4 \left[4 + \frac{4}{3} \right] \\
&= 4 \cdot \frac{16}{3} \\
&= \frac{64}{3}
\end{aligned}$$

is a constant, independent of the position of X.

2013.3 Question 4

We notice

$$(z-\exp(i\theta))(z-\exp(-i\theta))=z^2-(\exp(i\theta)+\exp(-i\theta))z+1=z^2-2z\cos\theta+1.$$

The $2n$-th roots of -1 are z_r, where $r=0,1,\ldots,2n-1$,

$$z_r=\exp\left(i\left(\frac{\pi}{2n}+\frac{2r\pi}{2n}\right)\right)=\exp\left(i\pi\cdot\frac{1+2r}{2n}\right),$$

and hence

$$\begin{aligned}
z^{2n}+1&=\prod_{r=0}^{2n-1}(z-z_r)\\
&=\left[\prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2r}{2n}\right)\right)\right]\cdot\left[\prod_{r=n}^{2n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2r}{2n}\right)\right)\right]\\
&=\left[\prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2r}{2n}\right)\right)\right]\cdot\left[\prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2(2n-1-r)}{2n}\right)\right)\right]\\
&=\left[\prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2r}{2n}\right)\right)\right]\cdot\left[\prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{-1-2r}{2n}\right)\right)\right]\\
&=\prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2r}{2n}\right)\right)\left(z-\exp\left(i\pi\cdot\frac{-1-2r}{2n}\right)\right)\\
&=\prod_{r=0}^{n-1}\left(z^2-2z\cos\left(\frac{2r+1}{2n}\pi\right)+1\right)\\
&=\prod_{r=1}^{n}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right).
\end{aligned}$$

1. Let $z=i$, since n is even, $z^{2n}=i^{2n}=(i^2)^n=(-1)^n=1$.

$$\begin{aligned}
2&=z^{2n}+1\\
&=\prod_{r=1}^{n}\left(i^2-2i\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)\\
&=\prod_{r=1}^{n}2i\cos\left(\frac{2r-1}{2n}\pi\right)\\
&=(2i)^n\prod_{r=1}^{n}\cos\left(\frac{2r-1}{2n}\pi\right)\\
&=2^n(-1)^{\frac{n}{2}}\prod_{r=1}^{n}\cos\left(\frac{2r-1}{2n}\pi\right),
\end{aligned}$$

   and therefore

$$\prod_{r=1}^{n}\cos\left(\frac{2r-1}{2n}\pi\right)=2^{1-n}(-1)^{-\frac{n}{2}}=2^{1-n}(-1)^{\frac{n}{2}}.$$

2. Notice that in the product where n is odd, let $k=\frac{n+1}{2}$, then the term of this product will be

$$\begin{aligned}
z^2-2z\cos\left(\frac{(2k-1)\pi}{2n}\right)+1&=z^2-2z\cos\left(\frac{(n+1-1)\pi}{2n}\right)+1\\
&=z^2-2z\cos\frac{\pi}{2}+1\\
&=z^2+1.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}(z^2+1)\sum_{r=0}^{n-1}(-1)^r z^{2r} &= z^2+1 \\ &= \prod_{r=1}^{n}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right) \\ &= \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)(z^2+1) \\ &\quad \prod_{r=\frac{n+3}{2}}^{n}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right) \\ &= \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)(z^2+1) \\ &\quad \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2(n+1-r)-1}{2n}\pi\right)+1\right),\end{aligned}$$

and hence

$$\begin{aligned}\sum_{r=0}^{n-1}(-1)^r z^{2r} &= \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)\left(z^2-2z\cos\left(\frac{2(n+1-r)-1}{2n}\pi\right)+1\right) \\ &= \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)\left(z^2-2z\cos\left(\frac{2n-2r+1}{2n}\pi\right)+1\right) \\ &= \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)\left(z^2+2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right).\end{aligned}$$

Let $z=i$, we have

$$\begin{aligned}\text{LHS} &= \sum_{r=0}^{n-1}(-1)^r i^{2r} \\ &= \sum_{r=0}^{n-1}(-1)^r (i^2)^r \\ &= \sum_{r=0}^{n-1}(-1)^r(-1)^r \\ &= \sum_{r=0}^{n-1}[(-1)(-1)]^r \\ &= \sum_{r=0}^{n-1}1 \\ &= n,\end{aligned}$$

and

$$\begin{aligned}
\text{RHS} &= \prod_{r=1}^{\frac{n-1}{2}} \left(i^2 - 2i\cos\left(\frac{2r-1}{2n}\pi\right) + 1\right)\left(i^2 + 2i\cos\left(\frac{2r-1}{2n}\pi\right) + 1\right) \\
&= \prod_{r=1}^{\frac{n-1}{2}} (-2i\cos\left(\frac{2r-1}{2n}\pi\right))(2i\cos\left(\frac{2r-1}{2n}\pi\right)) \\
&= \prod_{r=1}^{\frac{n-1}{2}} 4\cos^2\left(\frac{2r-1}{2n}\pi\right) \\
&= 2^{n-1}\prod_{r=1}^{\frac{n-1}{2}} \cos^2\left(\frac{2r-1}{2n}\pi\right).
\end{aligned}$$

This gives

$$\prod_{r=1}^{\frac{n-1}{2}} \cos^2\left(\frac{2r-1}{2n}\pi\right) = n2^{1-n},$$

exactly as desired.

2013.3 Question 5

1. Since $q^n N = p^n$, we have $p^n \mid q^n N$, and hence $p \mid q^n N$.

   But since $\gcd(p, q) = 1$, we must have $p \mid q^{n-1} N$. Repeating this step we will get $p \mid N$.

   Let $N = pN_1$, we have $q^n pN_1 = p^n$, giving $q^n N_1 = p^{n-1}$. Repeating the same step will give $p \mid N_1$.

   Let $N_1 = pN_2$, we have $q^n pN_2 = p^{n-1}$, giving $q^n N_2 = p^{n-2}$. Repeating the same step will give $p \mid N_2$.

   We can repeat this until we reach $q^n N_{n-1} = p$ from which we can conclude $p \mid N_{n-1}$.

   So $N_{n-1} = kp$ for some $k \in \mathbb{N}$.

   But since $N_t = pN_{t+1}$, we can conclude that $N_1 = kp^{n-1}$ and hence

   $$N = pN_1 = kp^n$$

   as desired.

   Hence, we have $q^n kp^n = p^n$ which gives $q^n k = 1$. B33gut this means q^n and k must both be one since $q, k \in \mathbb{N}$. Hence, $q = 1$.

   Assume, for the sake of contradiction, that $\sqrt[n]{N}$ is a rational number that is not a positive integer. Let

   $$\sqrt[n]{N} = \frac{p}{q},$$

   where $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$, and $q \neq 1$ (this is to ensure it is not a positive integer).

   Hence, by rearrangement, we have

   $$q^n N = p^n,$$

   and from what we have proved we must have $q = 1$, which contradicts with $q \neq 1$.

   Hence, $\sqrt[n]{N}$ must either be a positive integer or must be irrational.

2. Since $a^a d^b = b^a c^b$, we know that $a^a \mid b^a c^b$. By the same reasoning as part 1, we know that $c^b = ka^a$ for some positive integer k_1.

   Hence, putting it back to the original equation, we have

   $$d^b = k_1 b^a,$$

   which implies $d^b \geq b^a$.

   Since $a^a d^b = b^a c^b$, we know that $c^b \mid a^a d^b$. By the same reasoning as part 1, we know that $a^a = k_2 c^b$ for some positive integer k_2.

   Hence, putting it back to the original equation, we have

   $$k_2 d^b = b^a,$$

   which implies $b^a \geq d^b$.

   This means $d^b = b^a$.

   If a prime $p \mid d$, then $p \mid d^b$, and hence $p \mid b^a$.

   Since $b^a = bb^{a-1}$, if p does not divide b, this means p and b must be co-prime (since p is a prime), then p must divide b^{a-1}, and repeating this argument eventually reaches p dividing $b^{a-(a-1)}$ which is a contradiction. So p must divide b.

   Let $d = p^m d'$, and we must have p not divide d'. Similarly, let $b = p^n b'$, and we must have p does not divide b'.

   Putting this back to $d^b = b^a$ shows

   $$(p^m d')^b = (p^n b')^a,$$

   and hence

   $$p^{mb} d'^b = p^{na} b'^a,$$

   and we must have p does not divide d'^b nor b'^a.

This means p^{mb} and p^{na} are exactly the highest powers of p that divide $d^b = b^a$, and hence

$$mb = na \iff b = \frac{na}{m}.$$

Since $p^n \mid b$, we must have $p^n \mid \frac{na}{m}$, and hence $p^n \mid na$. However, since a and b are co-prime, and p is a prime factor of b, then p must not divide a, and hence $p^n \mid n$. Hence, $p^n \leq n$.

Since $y^x > x$ for $y \geq 2$ and $x > 0$, and $p^n \leq n$, we must have $p < 2$ or $n \leq 0$. But since p is a prime, $p \geq 2$, so we must have $n \leq 0$ and hence $n = 0$.

This means that the highest power of the prime number p that divides b is always 0, and hence $b = 1$.

Let

$$r = \frac{p}{q},$$

where $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$.

We have

$$r^r = \frac{r}{s}$$

for $r, s \in \mathbb{N}$, $\gcd(r, s) = 1$.

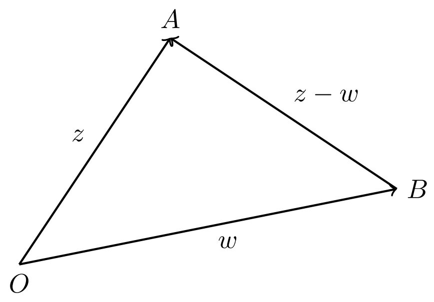
We have

$$\begin{aligned}
\left(\frac{p}{q}\right)^{\frac{p}{q}} &= \frac{r}{s} \\
\left(\frac{p}{q}\right)^{p} &= \left(\frac{r}{s}\right)^{q} \\
p^p s^q &= q^p r^q.
\end{aligned}$$

Here, let $a = p, b = q, c = r$ and $d = s$. We must have $b = q = 1$, which contradicts with $q \neq 1$.

Therefore, $r = p \in \mathbb{N}$ is a positive integer.

2013.3 Question 6



In the diagram, due to the triangular inequality, we must have $AB \leq OA + OB$, and hence $|z - w| \leq |z| + |w|$ as desired.

1. We have

$$\begin{aligned} \text{LHS} &= |z-w|^2 \\ &= (z-w)(z-w)^* \\ &= (z-w)(z^*-w^*) \\ &= zz^* + ww^* - zw^* - z^*w \\ &= |z|^2 + |w|^2 - (E - 2|zw|) \\ &= |z|^2 + 2|z||w| + |w|^2 - E \\ &= (|z| + |w|)^2 - E \\ &= \text{RHS}, \end{aligned}$$

exactly as desired.

Since $|z-w|$, $|z|$ and $|w|$ are all real, so must be $|z-w|^2$ and $(|z|+|w|)^2$, and so E must be real. Furthermore, we have

$$E = (|z| + |w|)^2 - |z-w|^2,$$

and by the inequality $|z| + |w| \geq |z - w| \geq 0$, we can conclude

$$(|z| + |w|)^2 \geq |z-w|^2,$$

and hence E must be non-negative.

2. We have

$$\begin{aligned} \text{LHS} &= |1 - zw^*|^2 \\ &= (1 - zw^*)(1 - zw^*)^* \\ &= (1 - zw^*)(1 - z^*w) \\ &= 1 - z^*w - zw^* + zwz^*w^* \\ &= 1 - (E - 2|zw|) + zw(zw)^* \\ &= 1 - (E - 2|zw|) + |zw|^2 \\ &= 1 + 2|zw| + |zw|^2 - E \\ &= (1 + |zw|)^2 - E \\ &= \text{RHS}. \end{aligned}$$

If we square both sides of the desired inequality (since both sides are non-negative this is reversible), we have

$$\frac{|z-w|^2}{|1-zw^*|^2} \leq \frac{(|z|+|w|)^2}{(1+|zw|)^2},$$

which is equivalent to showing

$$\frac{(|z|+|w|)^2-E}{(1+|zw|)^2-E} \leq \frac{(|z|+|w|)^2}{(1+|zw|)^2}.$$

We introduce a lemma. If $a > c \geq 0$ and $a > b$, then

$$\frac{b-c}{a-c} \leq \frac{b}{a}.$$

The proof of this is as follows. We cross-multiply the inequality to give (since $a \geq a-c > 0$ this is reversible)

$$a(b-c) \leq b(a-c),$$

which is equivalent to

$$ac \geq bc,$$

and this must be true given $c \geq 0$ and $a > b$.

Now, since $|z| > 1$, $|w| > 1$, we have

$$(|z|-1)(|w|-1) = 1+|zw|-|z|-|w| > 0,$$

which means

$$1+|zw| > |z|+|w|,$$

and since both are non-negative we have

$$(1+|zw|)^2 > (|z|+|w|)^2.$$

Now, using this lemma, let $a = (1+|zw|)^2, b = (|z|+|w|)^2, c = E$. $a > b$ is as shown in above, and $c \geq 0$ is shown in part 1. $a > c$ since $a - c = |1-zw^*|^2 \geq 0$, and the equal sign holds if and only if $|zw^*| = |zw| = 1$, which must not hold if $|z| > 1$ and $|w| > 1$ since this gives $|zw| = |z||w| > 1$.

Therefore, we must have

$$\frac{(|z|+|w|)^2-E}{(1+|zw|)^2-E} \leq \frac{(|z|+|w|)^2}{(1+|zw|)^2},$$

which gives exactly what is desired.

This also holds for $|z| < 1$ and $|w| < 1$ since from this $(|z|-1)(|w|-1) > 0$ still holds, so $(1+|zw|)^2 > (|z|+|w|)^2$ remains true, and $|zw| = |z||w| < 1$ so $|zw| \neq 1$ remains true. The exact argument is still valid.

2013.3 Question 7

1. We notice that

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}x} &= 2\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2y^3\frac{\mathrm{d}y}{\mathrm{d}x}\\ &= 2\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y^3\right)\\ &= 0,\end{aligned}$$

and so E must be constant.

So hence

$$\begin{aligned}E(x) &= E(0)\\ &= 0^2+\frac{1}{2}\\ &= \frac{1}{2}.\end{aligned}$$

Therefore,

$$y^4 = 2\left[E(x)-\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]\leq 2E(x)=1,$$

and hence

$$|y(x)|\leq 1.$$

2. We notice that

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}x} &= 2\cdot\frac{\mathrm{d}v}{\mathrm{d}x}\cdot\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+2\sinh v\frac{\mathrm{d}v}{\mathrm{d}x}\\ &= 2\frac{\mathrm{d}v}{\mathrm{d}x}\cdot\left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+\sinh v\right)\\ &= 2\frac{\mathrm{d}v}{\mathrm{d}x}\cdot\left(-x\frac{\mathrm{d}v}{\mathrm{d}x}\right)\\ &= -2x\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2,\end{aligned}$$

so when $x\geq 0$, since $\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2\geq 0$, we must have

$$\frac{\mathrm{d}E}{\mathrm{d}x}\leq 0.$$

Therefore, for $x\geq 0$, $E(x)\leq E(0)=0^2+2\cosh\ln 3=3+\frac{1}{3}=\frac{10}{3}$. Hence,

$$\begin{aligned}\cosh v(x) &= \frac{E(x)-\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2}{2}\\ &\leq \frac{\frac{10}{3}}{2}\\ &= \frac{5}{3}.\end{aligned}$$

3. Notice that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 &= 2\cdot\frac{\mathrm{d}v}{\mathrm{d}x}\cdot\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\\ &= -2\cdot\frac{\mathrm{d}w}{\mathrm{d}x}\cdot\left[(5\cosh x-4\sinh x-3)\cdot\frac{\mathrm{d}w}{\mathrm{d}x}+(w\cosh w+2\sinh w)\right].\end{aligned}$$

We also notice that

$$\begin{aligned}\int (w\cosh w + 2\sinh w)\,\mathrm{d}w &= \int w\cosh w\,\mathrm{d}w + 2\cosh w \\ &= \int w\,\mathrm{d}\sinh w + 2\cosh w + C \\ &= w\sinh w - \int \sinh w\,\mathrm{d}w + 2\cosh w + C \\ &= w\sinh w - \cosh w + 2\cosh w + C \\ &= w\sinh w + \cosh w + C,\end{aligned}$$

so consider the function

$$E(x) = \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + 2(w\sinh w + \cosh w),$$

and we have

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}x} &= -2 \cdot \frac{\mathrm{d}w}{\mathrm{d}x} \cdot \left[(5\cosh x - 4\sinh x - 3) \cdot \frac{\mathrm{d}w}{\mathrm{d}x} + (w\cosh w + 2\sinh w) - (w\cosh w + 2\sinh w)\right] \\ &= -2\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 (5\cosh x - 4\sinh x - 3) \\ &= -\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \left[5\left(e^x + e^{-x}\right) - 4\left(e^x - e^{-x}\right) - 6\right] \\ &= -\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \left(e^x + 9e^{-x} - 6\right) \\ &= -e^{-x}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 (e^x - 3)^2 \\ &\leq 0.\end{aligned}$$

Hence,

$$E(x) \leq E(0) = \left(\frac{1}{\sqrt{2}}\right)^2 + 2(0\sinh 0 + \cosh 0) = \frac{1}{2} + 2 = \frac{5}{2},$$

for $x \geq 0$.

Therefore,

$$\frac{5}{2} \geq \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + 2(w\sinh w + \cosh w),$$

and hence

$$2(w\sinh w + \cosh w) \leq \frac{5}{2}$$

for $x \geq 0$ since squares are always non-negative.

Hence,

$$\cosh w \leq \frac{5}{4} - w\sinh w \leq \frac{5}{4}$$

for $x \geq 0$, the second inequality being true since $w\sinh w \geq 0$ since $\sinh w$ and w always take the same sign, as desired.

2013.3 Question 8

By the formula of the sum for a geometric series, we have

$$\begin{aligned}\sum_{r=0}^{n-1} \exp(2i(\alpha + r\pi/n)) &= \exp(2i(\alpha + 0\pi/n)) \cdot \frac{1 - \exp(2i\pi/n)^n}{1 - \exp(2i\pi/n)} \\ &= \exp(2i\alpha) \cdot \frac{1 - \exp(2i\pi)}{1 - \exp(2i\pi/n)} \\ &= \exp(2i\alpha) \cdot \frac{1 - 1}{1 - \exp(2i\pi/n)} \\ &= 0,\end{aligned}$$

since the denominator is not 0.

By geometry, we have

$$r\cos\theta + s = d,$$

and hence

$$s = d - r\cos\theta.$$

Since $r = ks = k(d - r\cos\theta)$, we have

$$r = \frac{kd}{1 + k\cos\theta}.$$

Let L_1 be an angle α to horizontal, then L_j is angle $\alpha + (j-1)\pi/n$ angle to the horizontal for $j = 1, 2, \ldots, n$. Let $\theta_j = \alpha + (j-1)\pi/n$, and we have

$$\begin{aligned}l_j &= r|_{\theta=\theta_j} + r|_{\theta=\theta_j+\pi} \\ &= kd\left(\frac{1}{1 + k\cos\theta_j} + \frac{1}{1 + k\cos(\theta_j + \pi)}\right) \\ &= kd\left(\frac{1}{1 + k\cos\theta_j} + \frac{1}{1 - k\cos\theta_j}\right) \\ &= kd \cdot \frac{1 + k\cos\theta_j + 1 - k\cos\theta_j}{1 - k^2\cos^2\theta_j} \\ &= \frac{2kd}{1 - k^2\cos^2\theta_j}.\end{aligned}$$

Hence, we have

$$\begin{aligned}
\sum_{j=1}^{n} \frac{1}{l_j} &= \frac{1}{2kd} \sum_{j=1}^{n} (1 - k^2 \cos^2 \theta_j) \\
&= \frac{1}{2kd} \left[n - k^2 \sum_{j=1}^{n} \cos^2 (\alpha + (j-1)\pi/n) \right] \\
&= \frac{1}{2kd} \left[n - \frac{k^2}{2} \cdot \sum_{j=1}^{n} [1 + \cos 2 (\alpha + (j-1)\pi/n)] \right] \\
&= \frac{1}{2kd} \left[n - \frac{nk^2}{2} - \frac{k^2}{2} \cdot \sum_{j=1}^{n} \cos 2 (\alpha + (j-1)\pi/n) \right] \\
&= \frac{1}{2kd} \left[n - \frac{nk^2}{2} - \frac{k^2}{2} \cdot \sum_{r=0}^{n-1} \cos 2 (\alpha + r\pi/n) \right] \\
&= \frac{1}{2kd} \left[n - \frac{nk^2}{2} - \frac{k^2}{2} \cdot \sum_{r=0}^{n-1} \operatorname{Re} \exp(2i (\alpha + r\pi/n)) \right] \\
&= \frac{1}{2kd} \left[n - \frac{nk^2}{2} - \frac{k^2}{2} \cdot 0 \right] \\
&= \frac{1}{2kd} \cdot \frac{n(2-k^2)}{2} \\
&= \frac{n(2-k^2)}{4kd},
\end{aligned}$$

as desired.

2013.3 Question 12

1. Since $X_i \in \{0, 1\}$, we have $\mathrm{E}(X_i) = 0\,\mathrm{P}(X_i = 0) + 1\,\mathrm{P}(X_i = 1) = \mathrm{P}(X_i = 1)$.

   The total number of arrangements is
   $$\frac{n!}{a!b!}.$$

   To make $X_1 = 1$, we must have the first letter being A, and the rest can arrange to be whatever possible. Hence, the number of valid arrangements is
   $$\frac{(n-1)!}{(a-1)!b!}.$$

   Hence,
   $$\mathrm{E}(X_1) = v\,\mathrm{P}(X_1 = 1) = \frac{\frac{(n-1)!}{(a-1)!b!}}{\frac{n!}{a!b!}} = \frac{a}{n}.$$

   When $i \neq 1$, we must have the $i-1$th letter being B and the ith letter being A, and the rest can arrange to be whatever possible. Since $i > 1$, the $i-1$th letter will always exist. Hence, the number of valid arrangements is
   $$\frac{(n-2)!}{(a-1)!(b-1)!}$$

   Therefore,
   $$\mathrm{E}(X_i) = \mathrm{P}(X_i = 1) = \frac{\frac{(n-2)!}{(a-1)!(b-1)!}}{\frac{n!}{a!b!}} = \frac{ab}{n(n-1)}.$$

   Hence,
   $$\begin{aligned}\mathrm{E}(S) &= \mathrm{E}\left(\sum_{i=1}^{n} X_i\right) \\ &= \sum_{i=1}^{n} \mathrm{E}(X_i) \\ &= \frac{a}{n} + (n-1) \cdot \frac{ab}{n(n-1)} \\ &= \frac{a}{n} + \frac{ab}{n} \\ &= \frac{a(b+1)}{n}.\end{aligned}$$

2. (a) Notice that $X_1 X_j \in \{0, 1\}$, and $X_1 X_j = 1$ if and only if $X_1 = 1$ and $X_j = 1$. Hence,
   $$\mathrm{E}(X_1 X_j) = \mathrm{P}(X_1 = 1 \wedge X_j = 1).$$

   The arrangement for the event $X_1 = 1 \wedge X_j = 1$ must have the first letter A, the $j-1$-th letter B, and the j-th letter A. Since $j \geq 3$, we have $j - 1 \geq 2$ so will not repeat the requirement with the first letter. The rest can arrange whatever, so the number of valid arrangements is
   $$\frac{(n-3)!}{(a-2)!(b-1)!},$$

   and hence
   $$\mathrm{E}(X_1 X_j) = \mathrm{P}(X_1 = 1 \wedge X_j = 1) = \frac{\frac{(n-3)!}{(a-2)!(b-1)!}}{\frac{n!}{a!b!}} = \frac{a(a-1)b}{n(n-1)(n-2)},$$

   as desired.

(b) All terms in this sum satisfy $2 \leq i \leq n-2$ and $i+2 \leq j \leq n$. Notice that $X_i X_j \in \{0, 1\}$, and $X_i X_j = 1$ if and only if $X_i = 1$ and $X_j = 1$. Hence,

$$\mathrm{E}(X_i X_j) = \mathrm{P}(X_i = 1 \wedge X_j = 1).$$

The arrangement for the event $X_i = 1 \wedge X_j = 1$ must have the $i-1$-th letter B, i-th letter A, $j-1$-th letter B and j-th letter A. Since $j \geq i+2$, $j-1 \geq i+1 > i$, so the requirements do not repeat. Hence, the number of valid arrangements is

$$\frac{(n-4)!}{(a-2)!(b-2)!},$$

and hence

$$\mathrm{E}(X_i X_j) = \mathrm{P}(X_i = 1 \wedge X_j = 1) = \frac{\frac{(n-4)!}{(a-2)!(b-2)!}}{\frac{n!}{a!b!}} = \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}.$$

The number of terms in this sum is

$$\begin{aligned}
\sum_{i=2}^{n-2} \sum_{j=i+2}^{n} 1 &= \sum_{i=2}^{n-2} (n-(i+2)+1) \\
&= \sum_{i=2}^{n-2} (n-i-1) \\
&= [(n-2)-2+1](n-1) - \left[\frac{(n-2)(n-1)}{2} - 1\right] \\
&= (n-3)(n-1) - \left[\frac{n^2-3n}{2}\right] \\
&= (n-3)\left[(n-1) - \frac{n}{2}\right] \\
&= \frac{(n-3)(n-2)}{2}.
\end{aligned}$$

Hence, this sum evaluates to

$$\frac{(n-3)(n-2)}{2} \cdot \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)} = \frac{a(a-1)b(b-1)}{2n(n-1)},$$

exactly as desired.

(c) To find $\mathrm{Var}(S)$, we would like to find $\mathrm{E}(S^2)$. Notice that

$$\begin{aligned}
\mathrm{E}(S^2) &= \mathrm{E}\left(\left(\sum_{i=1}^{n} X_i\right)^2\right) \\
&= \mathrm{E}\left(\sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{E}(X_i X_j).
\end{aligned}$$

This sum can be further split up into these parts:

- Where $i = j$, the sum of $\mathrm{E}(X_i^2)$. But since X_i can only take 0 or 1, X_i^2 can only take 0 or 1, and we have

  $$\mathrm{P}(X_i = 0) = \mathrm{P}(X_i^2 = 0), \mathrm{P}(X_i = 1) = \mathrm{P}(X_i^2 = 1),$$

  and hence

  $$\mathrm{E}(X_i^2) = \mathrm{E}(X_i).$$

Hence, the sum can be evaluated as

$$\begin{aligned}\sum_{i=1}^{n} \mathrm{E}(X_i^2) &= \sum_{i=1}^{n} \mathrm{E}(X_i) \\ &= \mathrm{E}(X_1) + \sum_{i=2}^{n} \mathrm{E}(X_i) \\ &= \frac{a}{n} + (n-1) \cdot \frac{a(b+1)}{n(n-1)}.\end{aligned}$$

- Where $j = i \pm 1$. We can consider the case where $j = i + 1$ and double the result. For $X_i X_j = 1$, we must have $X_i = 1$ and $X_j = 1$, and hence the i-th letter must be A, and the $j - 1$-th letter must be B. But this is impossible since $j = i + 1$, and a letter cannot be both A and B. And hence

$$2 \cdot \sum_{i=1}^{n-1} \mathrm{E}(X_i X_{i+1}) = 0.$$

- Where $j \geq i + 2$ or $j \leq i - 2$. We consider the case where $j \geq i + 2$ and double the result. This is calculated in part a for the case $i = 1$, and part b for the case $i \geq 2$.

Hence,

$$\begin{aligned}\mathrm{E}(S^2) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{E}(X_i X_j) \\ &= \frac{a}{n} + (n-1) \cdot \frac{ab}{n(n-1)} + 2 \cdot \left[(n-2) \cdot \frac{a(a-1)b}{n(n-1)(n-2)} + \frac{a(a-1)b(b-1)}{2n(n-1)}\right] \\ &= \frac{a}{n} + \frac{ab}{n} + \frac{2a(a-1)b}{n(n-1)} + \frac{a(a-1)b(b-1)}{n(n-1)} \\ &= \frac{a(b+1)}{n} + \frac{a(a-1)b(b+1)}{n(n-1)} \\ &= \frac{a(b+1)}{n}\left[1 + \frac{(a-1)b}{n-1}\right].\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{Var}(S) &= \mathrm{E}(S^2) - \mathrm{E}(S)^2 \\ &= \frac{a(b+1)}{n}\left[1 + \frac{(a-1)b}{n-1}\right] - \left[\frac{a(b+1)}{n}\right]^2 \\ &= \frac{a(b+1)}{n}\left[1 + \frac{(a-1)b}{n-1} - \frac{a(b+1)}{n}\right] \\ &= \frac{a(b+1)}{n} \cdot \frac{n(n-1) + n(a-1)b - (n-1)a(b+1)}{n(n-1)} \\ &= \frac{a(b+1)}{n^2(n-1)}\left(n^2 - n + abn - nb - nab - na + ab + a\right) \\ &= \frac{a(b+1)}{n^2(n-1)}\left(n^2 - n - nb - na + ab + a\right) \\ &= \frac{a(b+1)}{n^2(n-1)}\left((a+b)^2 - (a+b) - (a+b)b - (a+b)a + ab + a\right) \\ &= \frac{a(b+1)}{n^2(n-1)}\left(a^2 + 2ab + b^2 - a - b - ab - b^2 - a^2 - ab + ab + a\right) \\ &= \frac{a(b+1)}{n^2(n-1)}(ab - b) \\ &= \frac{a(b+1)}{n^2(n-1)} b(a-1) \\ &= \frac{a(a-1)b(b+1)}{n^2(n-1)}.\end{aligned}$$

2013.3 Question 13

1. (a) Since $0 \le X \le 1$, we must have that

$$F(x) = \int_0^x f(t)\,\mathrm{d}t$$

for $0 \le x \le 1$. Hence, since $0 \le f(t) \le M$ for $0 \le t \le x \le 1$, we have

$$0 = \int_0^x 0\,\mathrm{d}t \le F(x) \le \int_0^x M\,\mathrm{d}t = Mx,$$

as desired.

(b) Since $0 \le X \le 1$, we must have $F(0) = 0$ and $F(1) = 1$. Let the desired integral be I, using integration by parts, we have

$$\begin{aligned}
I &= \int_0^1 2g(x)F(x)f(x)\,\mathrm{d}x \\
&= \int_0^1 2g(x)F(x)\,\mathrm{d}F(x) \\
&= \left[2g(x)F(x)^2\right]_0^1 - 2\int_0^1 F(x)\,\mathrm{d}(g(x)F(x)) \\
&= 2g(1)F(1)^2 - 2g(0)F(0)^2 - 2\int_0^1 g'(x)F(x)^2\,\mathrm{d}x - 2\int_0^1 g(x)F(x)f(x)\,\mathrm{d}x \\
&= 2g(1) - 2\int_0^1 g'(x)F(x)^2\,\mathrm{d}x - I.
\end{aligned}$$

This means

$$2I = 2g(1) - 2\int_0^1 g'(x)F(x)^2\,\mathrm{d}x,$$

and hence

$$I = g(1) - \int_0^1 g'(x)F(x)^2\,\mathrm{d}x.$$

2. (a) Since $0 \le Y \le 1$, we must have

$$\begin{aligned}
\int_0^1 kF(y)f(y)\,\mathrm{d}y &= k\int_0^1 F(y)\,\mathrm{d}F(y) \\
&= k \cdot \frac{1}{2} \cdot \left[F(y)^2\right]_0^1 \\
&= k \cdot \frac{1}{2} \cdot \left[F(1)^2 - F(0)^2\right] \\
&= \frac{k}{2} \cdot (1^2 - 0^2) \\
&= \frac{k}{2} \\
&= 1,
\end{aligned}$$

and hence $k = 2$.

(b) Notice that

$$\begin{aligned}
\mathrm{E}\left(Y^n\right) &= \int_0^1 2y^n F(y)f(y)\,\mathrm{d}y \\
&\le \int_0^1 2y^n My f(y)\,\mathrm{d}y \\
&= 2M\int_0^1 y^{n+1} f(y)\,\mathrm{d}y \\
&= 2M\,\mathrm{E}\left(X^{n+1}\right) \\
&= 2M\mu_{n+1},
\end{aligned}$$

and that

$$\begin{aligned}
\mathrm{E}\left(Y^n\right) &= \int_0^1 2y^n F(y) f(y)\,\mathrm{d}y \\
&= y^n\big|_{y=1} - \int_0^1 (y^n)' F(y)^2\,\mathrm{d}y \\
&= 1 - n\int_0^1 y^{n-1} F(y)^2\,\mathrm{d}y \\
&\geq 1 - n\int_0^1 y^{n-1} M y F(y)\,\mathrm{d}y \\
&= 1 - Mn\int_0^1 y^n F(y)\,\mathrm{d}y \\
&= 1 - \frac{Mn}{n+1}\int_0^1 F(y)\,\mathrm{d}(y^{n+1}) \\
&= 1 - \frac{Mn}{n+1}\left(\left[F(y)y^{n+1}\right]_0^1 - \int_0^1 y^{n+1}\,\mathrm{d}F(y)\right) \\
&= 1 - \frac{Mn}{n+1}\left(F(1)\cdot 1^{n+1} - F(0)\cdot 0^{n+1} - \int_0^1 y^{n+1} f(y)\,\mathrm{d}y\right) \\
&= 1 - \frac{Mn}{n+1}\left(1 - \mathrm{E}\left(X^{n+1}\right)\right) \\
&= 1 - \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1},
\end{aligned}$$

as desired.

(c) Since we have for non-negative n,

$$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \leq 2M\mu_{n+1},$$

and hence for $n \geq 1$, we have

$$1 + \frac{(n-1)M}{n}\mu_n - \frac{(n-1)M}{n} \leq 2M\mu_n,$$

which multiplying both sides by n gives

$$n + (n-1)M\mu_n - (n-1)M \leq 2Mn\mu_n,$$

and rearranging gives

$$n - (n-1)M \leq M(n+1)\mu_n,$$

hence

$$\mu_n \geq \frac{n-(n-1)M}{M(n+1)} = \frac{n}{(n+1)M} - \frac{n-1}{n+1},$$

as desired.