2012 Paper 3

2012.3 Question 1

We have

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= n \cdot y^{n-1} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y^n \cdot 2 \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\ &= y^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}\left[n\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right],\end{aligned}$$

as desired.

1. Let $n = 1$, we have $z = y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, and

   $$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right].$$

   Hence, the differential equation

   $$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{y}$$

   simplifies to

   $$\frac{\frac{\mathrm{d}z}{\mathrm{d}x}}{\frac{\mathrm{d}y}{\mathrm{d}x}} = \sqrt{y},$$

   and hence

   $$\frac{\mathrm{d}z}{\mathrm{d}y} = \sqrt{y}.$$

   Hence, by integration,

   $$z = \frac{2}{3}y^{\frac{3}{2}} + C.$$

   When $x = 0$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, and hence $z = 0$. Hence,

   $$0 = \frac{2}{3} + C,$$

   and therefore $C = -\frac{2}{3}$.
   We therefore have

   $$y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{2}{3}y^{\frac{3}{2}} - \frac{2}{3},$$

   and hence

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{2}{3}\left(\sqrt{y} - \frac{1}{y}\right)}.$$

   Rearrangement gives

   $$\frac{\sqrt{y}\,\mathrm{d}y}{\sqrt{y^{\frac{3}{2}} - 1}} = \sqrt{\frac{2}{3}}\,\mathrm{d}x.$$

   Notice that

   $$\begin{aligned}\frac{\mathrm{d}\sqrt{y^{\frac{3}{2}} - 1}}{\mathrm{d}y} &= \frac{1}{2} \cdot \frac{1}{\sqrt{y^{\frac{3}{2}} - 1}} \cdot \frac{3}{2} \cdot \sqrt{y} \\ &= \frac{3}{4} \cdot \frac{\sqrt{y}}{\sqrt{y^{\frac{3}{2}} - 1}},\end{aligned}$$

and hence by integration

$$\frac{4}{3} \cdot \sqrt{y^{\frac{3}{2}} - 1} = \sqrt{\frac{2}{3}}x + C.$$

When $x = 0, y = 1$, and hence $C = 0$. Therefore,

$$\sqrt{y^{\frac{3}{2}} - 1} = \sqrt{\frac{3}{8}}x,$$

and hence

$$y^{\frac{3}{2}} = \frac{3}{8}x^2 + 1,$$

and hence

$$y = \left(\frac{3}{8}x^2 + 1\right)^{\frac{2}{3}},$$

as desired.

2. Let $n = -2$, we have $z = y^{-2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, and

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right].$$

Hence, the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^2 = 0$$

simplifies to

$$\frac{\frac{\mathrm{d}z}{\mathrm{d}x}}{-2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}} + y^2 = 0,$$

which gives

$$\frac{\mathrm{d}z}{\mathrm{d}y} = \frac{2}{y}.$$

By integration on both sides, we have

$$z = 2\ln y + C,$$

and when $x = 0$, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, which gives $z = 0$. Hence, $C = 0$, and

$$y^{-2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 2\ln y,$$

which gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\sqrt{2\ln y},$$

and therefore,

$$\frac{\mathrm{d}y}{y\sqrt{\ln y}} = \sqrt{2}\,\mathrm{d}x.$$

By integration,

$$\int \frac{\mathrm{d}y}{y\sqrt{\ln y}} = \int \frac{\mathrm{d}\ln y}{\sqrt{\ln y}} = 2\sqrt{\ln y} + C,$$

and hence

$$2\sqrt{\ln y} = \sqrt{2}x + C.$$

When $x = 0$, $y = 1$, so $C = 0$, and hence

$$\sqrt{\ln y} = \frac{x}{\sqrt{2}},$$

and therefore, the solution to the original differential equation is

$$y = e^{\frac{x^2}{2}}.$$

2012.3 Question 2

1. By the formula for difference of two squares, we have

$$\begin{aligned}(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) &= (1-x^2)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) \\ &= (1-x^4)(1+x^4)\cdots(1+x^{2^n}) \\ &= \cdots \\ &= 1-x^{2^{n+1}}.\end{aligned}$$

This means,

$$1 = (1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) + x^{2^{n+1}},$$

and dividing both sides by $1-x$ gives

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) + \frac{x^{2^{n+1}}}{1-x}.$$

Rearranging and taking natural logs on both sides, we have

$$\ln(1-x^{2^{n+1}}) - \ln(1-x) = \sum_{k=0}^{n} \ln(1+x^{2^k}),$$

and therefore,

$$\ln(1-x) = -\sum_{k=0}^{n} \ln(1+x^{2^k}) + \ln(1-x^{2^{n+1}}).$$

Let $n\to\infty$. $2^{n+1}\to\infty$, and since $|x|<1$, we have $x^{2^{n+1}}\to 0$, and hence

$$\ln(1-x) = -\sum_{k=0}^{\infty} \ln(1+x^{2^k}) + \ln(1) = -\sum_{k=0}^{\infty} \ln(1+x^{2^k}),$$

as desired.

Differentiating both sides with respect to x, we have

$$-\frac{1}{1-x} = -\sum_{k=0}^{\infty} \frac{2^k x^{2^k-1}}{1+x^{2^k}},$$

and hence

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} \frac{2^k x^{2^k-1}}{1+x^{2^k}},$$

exactly as desired.

2. Notice that

$$\begin{aligned}&(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n}) \\ &= ((1+x^2)^2-x^2)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n}) \\ &= (1+x^2+x^4)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n}) \\ &= ((1+x^4)^2-(x^2)^2)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n}) \\ &= (1+x^4+x^8)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n}) \\ &= \cdots \\ &= 1+x^{2^n}+x^{2^{n+1}}.\end{aligned}$$

Therefore,

$$1 = (1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n}) - x^{2^n} - x^{2^{n+1}},$$

and hence

$$\frac{1}{1+x+x^2} = (1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n}) - \frac{x^{2^n}+x^{2^{n+1}}}{1+x+x^2}.$$

Rearranging and taking natural logs on both sides, we have

$$\ln(1+x^{2^n}+x^{2^{n+1}}) - \ln(1+x+x^2) = \sum_{k=1}^{n} \ln(1-x^{2^{k-1}}+x^{2^k}),$$

and hence

$$\ln(1+x+x^2) = -\sum_{k=1}^{n} \ln(1-x^{2^{k-1}}+x^{2^k}) + \ln(1+x^{2^n}+x^{2^{n+1}}).$$

Let $n \to \infty$, we have $2^n, 2^{n+1} \to \infty$, and since $|x| < 1$, we must have $x^{2^n}, x^{2^{n+1}} \to \infty$, and hence $\ln(1+x^{2^n}+x^{2^{n+1}}) \to 0$. Hence,

$$\ln(1+x+x^2) = -\sum_{k=1}^{\infty} \ln(1-x^{2^{k-1}}+x^{2^k}).$$

Differentiating both sides with respect to x, we get

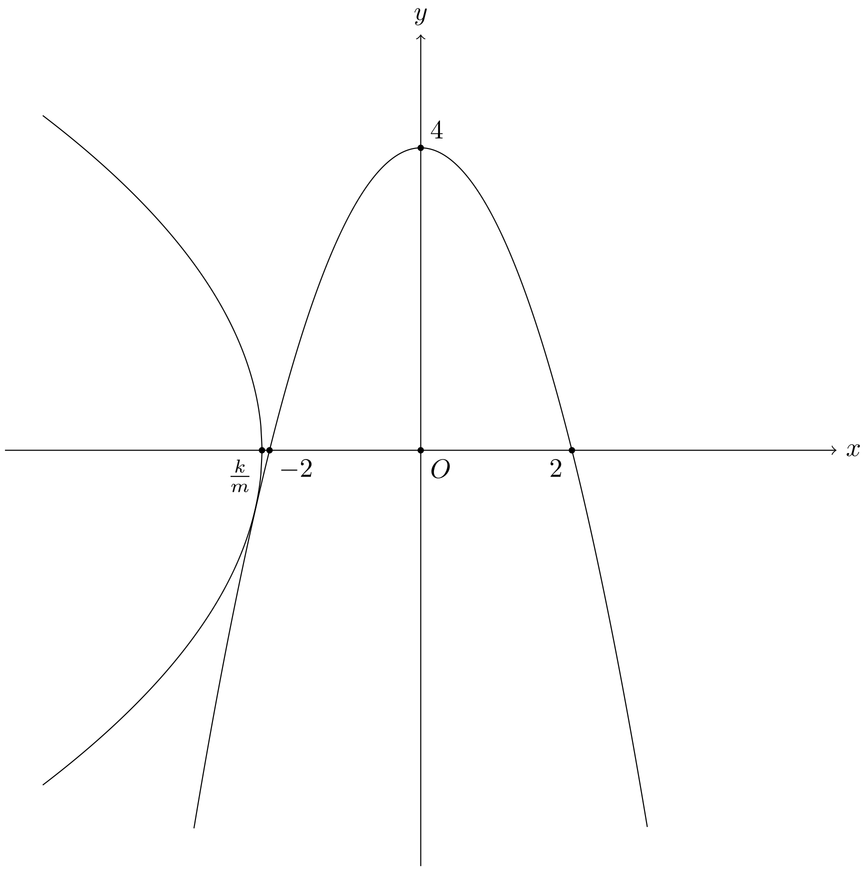
$$\frac{1+2x}{1+x+x^2} = -\sum_{k=1}^{\infty} \frac{-2^{k-1}x^{2^{k-1}-1}+2^k x^{2^k-1}}{1-x^{2^{k-1}}+x^{2^k}} = \sum_{k=1}^{\infty} \frac{2^{k-1}x^{2^{k-1}-1}-2^k x^{2^k-1}}{1-x^{2^{k-1}}+x^{2^k}},$$

which is exactly what is desired.

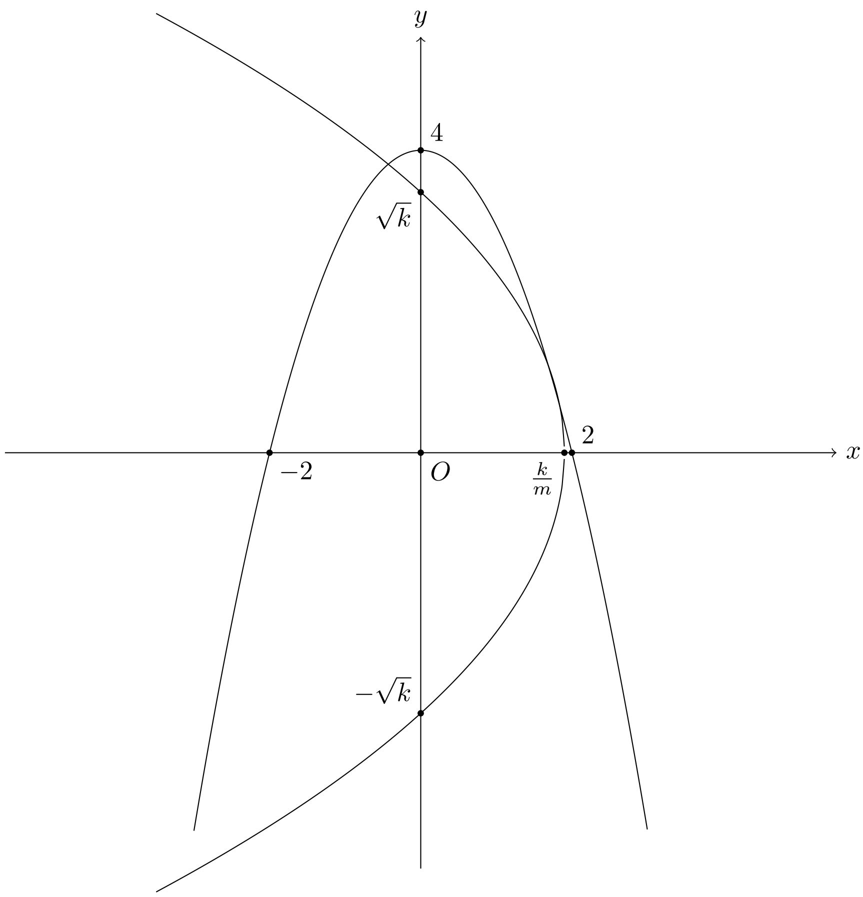
2012.3 Question 3

1. Let the two curves be $\Gamma_1 : y = 4 - x^2$ and $\Gamma_2 : x = -\frac{y^2}{m} + \frac{k}{m}$.
   For the first curve, its y-intercept is 4, and its x-intercept is ± 2.
   For the second curve, its y-intercept is $\pm\sqrt{k}$ (if $k \geq 0$), and its x-intercept is $\frac{k}{m}$.

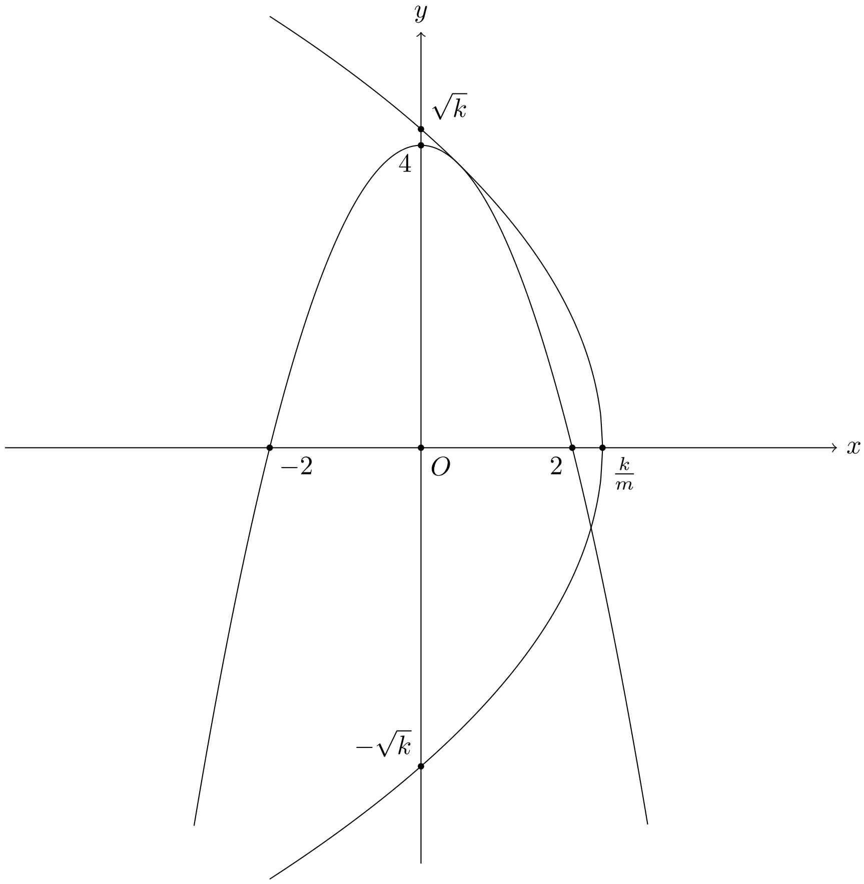
   (a) Since $k < 0$, we must have $\frac{k}{m} < 0$ as well, and hence the curves must look as follows:



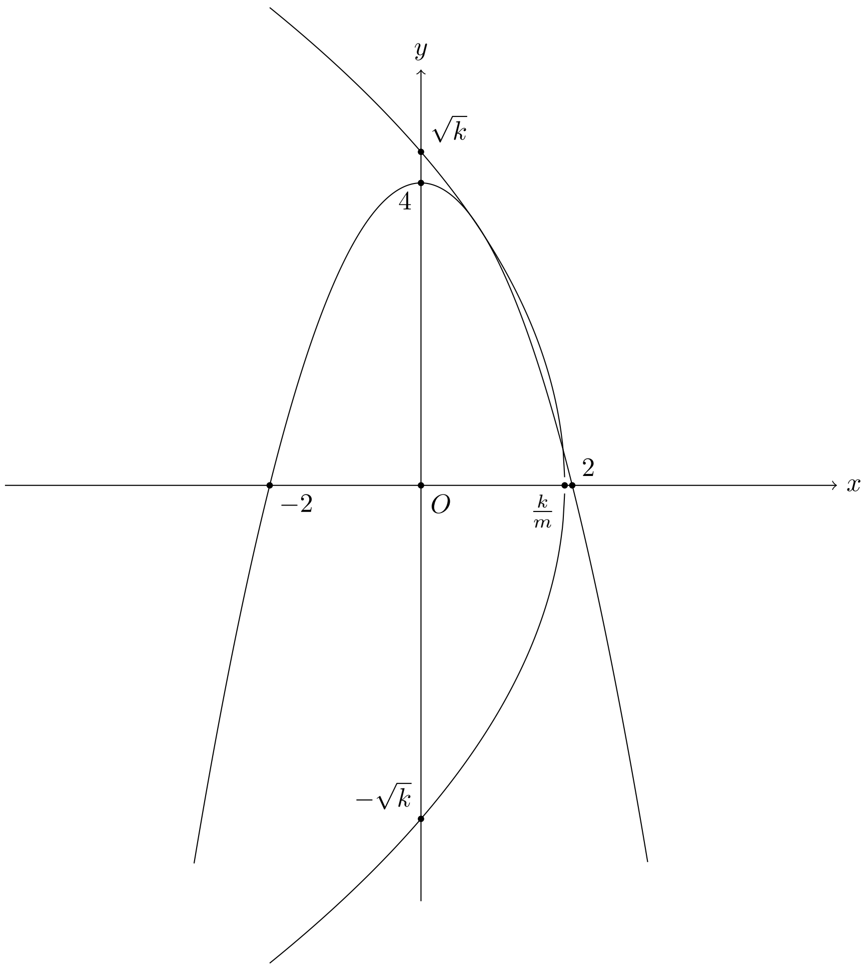
   (b) Since $0 < k < 16$, Γ_2 must have a y-intercept less than that of Γ_1. Since $\frac{k}{m} < 2$, Γ_2 must have the x-intercept to the left of $(2, 0)$. Hence, the curves must look as follows:



(c) Since $k > 16$, Γ_2 must have a y-intercept greater than that of Γ_1. Since $\frac{k}{m} > 2$, Γ_2 must have the x-intercept to the right of $(2, 0)$. Hence, the curves must look as follows:



(d) Since $k > 16$, Γ_2 must have a y-intercept greater than that of Γ_1. Since $\frac{k}{m} < 2$, Γ_2 must have the x-intercept to the left of $(2, 0)$. Hence, the curves must look as follows:



2. Since $y = y$, we must have

$$12x = k - (4 - x^2)^2 = k - 16 + 8x^2 - x^4,$$

and hence

$$x^4 - 8x^2 + 12x + 16 - k = 0,$$

as desired.

For the first curve, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x,$$

and applying implicit differentiation on both sides of the second equation, we must have

$$12 = -2y\frac{\mathrm{d}y}{\mathrm{d}x},$$

and hence

$$12 = (-2y) \cdot (-2x),$$

which gives $xy = 3$ for the point where the curves touch.

Hence,

$$\frac{3}{a} = 4 - a^2,$$

and this gives

$$a^3 - 4a + 3 = 0$$

as desired.

Notice that

$$a^3 - 4a + 3 = (a - 1)(a^2 + a - 3),$$

and hence the three solutions to a are

$$a_1 = 1, a_{2,3} = \frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1 \pm \sqrt{13}}{2}.$$

From the first equation, we must have

$$\begin{aligned} k &= a^4 - 8a^2 + 12a + 16 \\ &= a(a^3 - 4a + 3) - 4a^2 + 9a + 16 \\ &= a \cdot 0 - 4a^2 + 9a + 16 \\ &= -4a^2 + 9a + 16, \end{aligned}$$

as desired.

For $a = 1$, $k = -4 \cdot 1^2 + 9 \cdot 1 + 16 = -4 + 9 + 16 = 21$, and $\frac{k}{m} = \frac{21}{12} < 2$, so (d) arises.

When $a_{2,3} = \frac{-1 \pm \sqrt{13}}{2}$, we have $a^2 + a - 3 = 0$, and hence

$$k = -4a^2 + 9a + 16 = -4(a^2 + a - 3) + 13a + 4 = 13a + 4.$$

For $a_2 = \frac{-1+\sqrt{13}}{2}$, we have

$$k = \frac{-13 + 13\sqrt{13}}{2} + 4 = \frac{-5 + 13\sqrt{13}}{2}.$$

Since $13\sqrt{13} > 13 \cdot 3 = 39$, we must have $-5 + 13\sqrt{13} > 34$, and hence $k > \frac{34}{2} = 17 > 16$.

We also have $13\sqrt{13} < 13 \cdot 4 = 52$, and hence $-5 + 13\sqrt{13} < 47$, and hence $k < \frac{47}{2}$, which means

$$\frac{k}{m} < \frac{47}{2 \cdot 12} = \frac{47}{24} < 2,$$

so case (d) arises.

For $a_3 = \frac{-1-\sqrt{13}}{2}$, we have $k = \frac{-13-13\sqrt{13}}{2} + 4 = \frac{-5-13\sqrt{13}}{4} < 0$, and so (a) arises.

2012.3 Question 4

1. Using the Maclaurin Expansion of e^x and setting $x = 1$, we have

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n+1}{n!} &= \sum_{n=1}^{\infty} \frac{n}{n!} + \sum_{n=1}^{\infty} \frac{1}{n!} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!} - \frac{1}{0!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} - 1 \\ &= e + e - 1 \\ &= 2e - 1.\end{aligned}$$

We have as well

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!} &= \sum_{n=1}^{\infty} \frac{n(n-1) + 3n + 1}{n!} \\ &= \sum_{n=1}^{\infty} \frac{n(n-1)}{n!} + 3\sum_{n=1}^{\infty} \frac{n}{n!} + \sum_{n=1}^{\infty} \frac{1}{n!} \\ &= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} + 3\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!} - 1 \\ &= 5\sum_{n=0}^{\infty} \frac{1}{n!} - 1 \\ &= 5e - 1,\end{aligned}$$

as desired.

We also have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(2n-1)^3}{n!} &= \sum_{n=1}^{\infty} \frac{8n^3 - 12n^2 + 6n - 1}{n!} \\ &= \sum_{n=1}^{\infty} \frac{8n(n-1)(n-2) + 12n(n-1) + 2n - 1}{n!} \\ &= 8\sum_{n=3}^{\infty} \frac{1}{(n-3)!} + 12\sum_{n=2}^{\infty} \frac{1}{(n-2)!} + 2\sum_{n=1}^{\infty} \frac{1}{(n-1)!} - \sum_{n=0}^{\infty} \frac{1}{n!} + 1 \\ &= (8 + 12 + 2 - 1)\sum_{n=0}^{\infty} \frac{1}{n!} + 1 \\ &= 21e + 1.\end{aligned}$$

2. Using the Maclaurin Expansion of $\ln(1-x)$ and letting $x = \frac{1}{2}$, we have

$$\ln 2 = -\ln\left(1 - \frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n} = \sum_{n=1}^{\infty} \frac{2^{-n}}{n}.$$

Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n^2+1)2^{-n}}{(n+1)(n+2)} &= \sum_{n=0}^{\infty} \frac{[(n+1)(n+2)-5(n+1)+2(n+2)]2^{-n}}{(n+1)(n+2)} \\
&= \sum_{n=0}^{\infty} 2^{-n} - 5\sum_{n=0}^{\infty} \frac{2^{-n}}{n+2} + 2\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1} \\
&= 2 - 5\cdot 4\sum_{n=2}^{\infty} \frac{2^{-n}}{n} + 2\cdot 2\sum_{n=1}^{\infty} \frac{2^{-n}}{n} \\
&= 2 - 20(\ln 2 - \frac{1}{2}) + 4(\ln 2) \\
&= -16\ln 2 + 12.
\end{aligned}$$

2012.3 Question 5

1. (a) An integer point: $(0,1)$. A non-integer point: $\left(\frac{3}{5},\frac{4}{5}\right)$.

   (b) An integer rational point: $(1,1)$. Notice that

$$\begin{aligned}&(\cos\theta+\sqrt{m}\sin\theta)^2+(\sin\theta-\sqrt{m}\cos\theta)^2\\&=\cos^2\theta+2\sqrt{m}\sin\theta\cos\theta+m\sin^2\theta+\sin^2\theta-2\sqrt{m}\sin\theta\cos\theta+m\cos^2\theta\\&=(m+1)(\sin^2\theta+\cos^2\theta)\\&=m+1.\end{aligned}$$

   Consider letting $x=\cos\theta+\sqrt{m}\sin\theta$, and $y=\sin\theta-\sqrt{m}\cos\theta$. Let $m=1$, and we have $x=\cos\theta+\sin\theta$ and $y=\sin\theta-\cos\theta$, with $x^2+y^2=m+1=2$.
   Let $\cos\theta=\frac{3}{5}$, and $\sin\theta=\frac{4}{5}$. We have

$$(x,y)=\left(\frac{7}{5},\frac{1}{5}\right)$$

   is a non-integer rational point.

2. (a) An integer 2-rational point: $(1,\sqrt{2})$.
   For the non-integer 2-rational point, let $m=\sqrt{2}$ in the previous question, and we have

$$(\cos\theta+\sqrt{2}\sin\theta)^2+(\sin\theta-\sqrt{2}\cos\theta)^2=2+1=3.$$

   Now, let $\cos\theta=\frac{3}{5}$ and $\sin\theta=\frac{4}{5}$. Let $x=\cos\theta+\sqrt{2}\sin\theta=\frac{3}{5}+\sqrt{2}\cdot\frac{4}{5}$ and $y=\sin\theta-\sqrt{2}\cos\theta=\frac{4}{5}-\sqrt{2}\cdot\frac{3}{5}$. We must have $x^2+y^2=3$, and

$$(x,y)=\left(\frac{3}{5}+\sqrt{2}\cdot\frac{4}{5},\frac{4}{5}-\sqrt{2}\cdot\frac{3}{5}\right)$$

   is a non-integer 2-rational point on $x^2+y^2=3$.

   (b) Consider $x=a\cos\theta+b\sqrt{m}\sin\theta$ and $y=a\sin\theta-b\sqrt{m}\cos\theta$, we have

$$\begin{aligned}x^2+y^2&=\left(a\cos\theta+b\sqrt{m}\sin\theta\right)^2+\left(a\sin\theta-b\sqrt{m}\cos\theta\right)^2\\&=a^2\cos^2\theta+b^2m\sin^2\theta+2ab\sqrt{m}\sin\theta\cos\theta\\&\quad+a^2\sin^2\theta+b^2m\cos^2\theta-2ab\sqrt{m}\sin\theta\cos\theta\\&=(a^2+b^2m)\cos^2\theta+(a^2+b^2m)\sin^2\theta\\&=(a^2+b^2m)(\sin^2\theta+\cos^2\theta)\\&=a^2+b^2m.\end{aligned}$$

   We set $m=2$, and hence we would like $a^2+2b^2=11$. Consider $a=3$ and $b=1$, and set $\cos\theta=\frac{4}{5}$ and $\sin\theta=\frac{3}{5}$. Hence,

$$x=a\cos\theta+b\sqrt{m}\sin\theta=3\cdot\frac{4}{5}+1\cdot\sqrt{2}\cdot\frac{3}{5}=\frac{12}{5}+\sqrt{2}\cdot\frac{3}{5},$$

   and

$$y=a\sin\theta-b\sqrt{m}\cos\theta=3\cdot\frac{3}{5}-1\cdot\sqrt{2}\cdot\frac{4}{5}=\frac{9}{5}-\sqrt{2}\cdot\frac{4}{5},$$

   and we must have $x^2+y^2=3^2+1^2\cdot 2=11$. Therefore,

$$(x,y)=\left(\frac{12}{5}+\sqrt{2}\cdot\frac{3}{5},\frac{9}{5}-\sqrt{2}\cdot\frac{4}{5}\right)$$

   is a non-integer 2-rational point on the circle $x^2+y^2=11$.

(c) Consider $x = a\sec\theta + b\sqrt{m}\tan\theta$ and $y = a\tan\theta + b\sqrt{m}\sec\theta$, we have

$$\begin{aligned} x^2 - y^2 &= \left(a\sec\theta + b\sqrt{m}\tan\theta\right)^2 - \left(a\tan\theta + b\sqrt{m}\sec\theta\right)^2 \\ &= a^2\sec^2\theta + b^2 m\tan^2\theta + 2ab\sqrt{m}\sec\theta\tan\theta \\ &\quad - a^2\tan^2\theta - b^2 m\sec^2\theta - 2ab\sqrt{m}\sec\theta\tan\theta \\ &= a^2(\sec^2\theta - \tan^2\theta) - b^2 m(\sec^2\theta - \tan^2\theta) \\ &= a^2 - b^2 m. \end{aligned}$$

We set $m = 2$, and hence we would like $a^2 - 2b^2 = 7$. Consider $a = 3$ and $b = 1$, and set $\tan\theta = \frac{3}{4}$ and $\sec\theta = \frac{5}{4}$. Hence,

$$x = a\sec\theta + b\sqrt{m}\tan\theta = 3\cdot\frac{5}{4} + 1\cdot\sqrt{2}\cdot\frac{3}{4} = \frac{15}{4} + \sqrt{2}\cdot\frac{3}{4},$$

and

$$y = a\tan\theta + b\sqrt{m}\sec\theta = 3\cdot\frac{3}{4} + 1\cdot\sqrt{2}\cdot\frac{5}{4} = \frac{9}{4} + \sqrt{2}\cdot\frac{5}{4},$$

and we must have $x^2 - y^2 = 3^2 - 1^2\cdot 2 = 7$. Therefore,

$$(x, y) = \left(\frac{15}{4} + \sqrt{2}\cdot\frac{3}{4}, \frac{9}{4} + \sqrt{2}\cdot\frac{5}{4}\right)$$

is a non-integer 2-rational point on the hyperbola $x^2 - y^2 = 7$.

2012.3 Question 6

Since $x + iy$ is a root of this quadratic equation, putting it back into the original equation, we have

$$(x + iy)^2 + p(x + iy) + 1 = (x^2 - y^2 + px + 1) + (2x + p)yi = 0,$$

and so it must have both real parts and complex parts 0, and hence $x^2 - y^2 + px + 1 = 0$, and $(2x + p)y = 0$.

Since $(2x + p)y = 0$, we must have either $2x + p = 0$ (which gives $p = -2x$), or $y = 0$. In the latter case, we put this back into the first equation, and we have

$$x^2 + px + 1 = 0.$$

If $x = 0$, then we must have $0 + 0 + 1 = 1 = 0$ which is impossible. Hence, $x \neq 0$, and by rearranging, we have

$$p = -\frac{x^2 + 1}{x}.$$

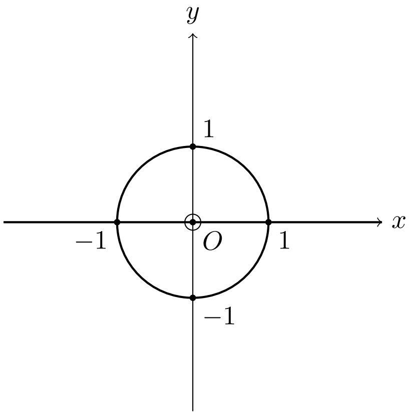
In the case where $p = -2x$, we must have

$$x^2 - y^2 + (-2x) \cdot x + 1 = 0 \iff x^2 + y^2 = 1,$$

and this represents a circle centred at the origin with radius 1.

In the case where $p = -\frac{x^2+1}{x}$, we must have $y = 0$, and $x \neq 0$. This represents the real axis without the origin.

This is the root locus of this equation.



For the second equation, let $z = x + iy$ be a solution. We have

$$p(x + iy)^2 + (x + iy) + 1 = (px^2 - py^2 + x + 1) + (2px + 1)yi = 0,$$

and so $px^2 - py^2 + x + 1 = 0$ and $(2px + 1)y = 0$.

Since $(2px + 1)y = 0$, we must have either $2px + 1 = 0$ (which gives $p = -\frac{1}{2x}$ since $x \neq 0$, or otherwise $0 + 1 = 1 = 0$), or $y = 0$. In the latter case, we put this back to the first equation, and we have

$$px^2 + x + 1 = 0.$$

If $x = 0$ then we must have $0 + 0 + 1 = 1 = 0$ which is impossible. Hence, $x \neq 0$, and by rearranging, we have

$$p = -\frac{x + 1}{x^2}.$$

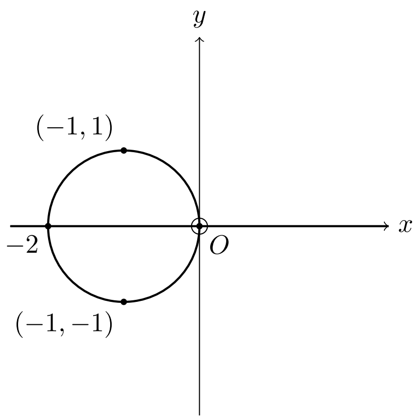
In the case where $p = -\frac{1}{2x}$, given $x \neq 0$,

$$-\frac{1}{2x}(x^2 - y^2) + x + 1 = 0 \iff \frac{x}{2} + \frac{y^2}{2x} + 1 = 0 \iff (x + 1)^2 + y^2 = 1.$$

This represents a circle centred at $(-1, 0)$ with radius 1, and since $x \neq 0$, we have to remove the point $(0, 0)$.

In the case where $p = -\frac{x+1}{x^2}$, $y = 0$ and this represents the real axis without the origin.

This is the root locus of this equation.



For the final equation, let $z = x + iy$ be a solution. We have

$$p(x+iy)^2 + p^2(x+iy) + 2 = (px^2 - py^2 + p^2x + 2) + yp(2x+p)i = 0,$$

and so $px^2 - py^2 + p^2x + 2 = 0$ and $yp(2x+p) = 0$.

Notice that here, $p \neq 0$, since if $p = 0$ then $2 = 0$ and there is no solution. So since $yp(2x+p) = 0$, we have $2x + p = 0$ which gives $p = -2x$, or $y = 0$. In the latter case, we put this back to the first equation, and we have

$$px^2 + p^2x + 2 = 0.$$

If $x = 0$ then we must have $0 + 0 + 2 = 2 = 0$ which is impossible. Hence, $x \neq 0$. For this to have a real solution for p, we must have $x \neq 0$ and

$$(x^2)^2 - 4 \cdot x \cdot 2 \geq 0,$$

which means

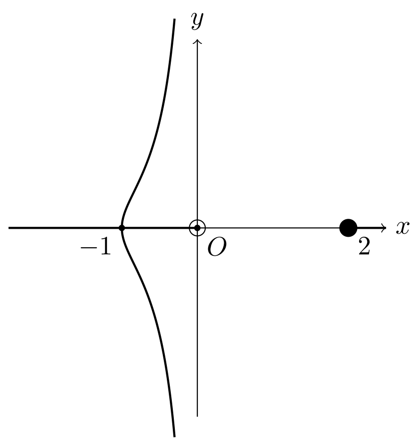
$$x(x-2)(x^2+2x+2) \geq 0.$$

Since $x^2 + 2x + 2 = (x+1)^2 + 1 \geq 1 \geq 0$, we must have $x(x-2) \geq 0$, and $x \leq 0$ or $x \geq 2$. This represents the real line with the interval $[0, 2)$ removed.

In the case where $p = -2x$, putting this back to the first equation, we have

$$(-2x)x^2 - (-2x)y^2 + (-2x)^2x + 2 = 0 \iff x^3 + xy^2 + 1 = 0 \iff y^2 = -\frac{1+x^3}{x}.$$

This is the root locus of this equation.



2012.3 Question 7

Since $\dot{y} = -2(y - z)$, differentiating both sides with respect to t gives

$$\begin{aligned}\ddot{y} &= -2\dot{y} + 2\dot{z} \\ &= -2\dot{y} + 2(-\dot{y} - 3z) \\ &= -4\dot{y} - 6z \\ &= -4\dot{y} - 3(\dot{y} + 2y) \\ &= -7\dot{y} - 6y,\end{aligned}$$

and hence

$$\ddot{y} + 7\dot{y} + 6y = 0.$$

The auxiliary equation

$$\lambda^2 + 7\lambda + 6 = 0$$

gives roots

$$\lambda_1 = -1, \lambda_2 = -6,$$

and hence

$$y = Ae^{-t} + Be^{-6t}.$$

Hence,

$$\dot{y} = -Ae^{-t} - 6Be^{-6t},$$

and therefore,

$$\begin{aligned}z &= \frac{\dot{y} + 2y}{2} \\ &= \frac{(-Ae^{-t} - 6Be^{-6t}) + 2(Ae^{-t} + Be^{-6t})}{2} \\ &= \frac{Ae^{-t} - 4Be^{-6t}}{2} \\ &= \frac{1}{2}Ae^{-t} - 2Be^{-6t}.\end{aligned}$$

This set of general solution

$$(y, z) = \left(Ae^{-t} + Be^{-6t}, \frac{1}{2}Ae^{-t} - 2Be^{-6t}\right),$$

is exactly what is desired.

1. $y(0) = 5$ and $z(0) = 0$ gives the system of linear equations

$$\begin{cases} A + B = 5, \\ \frac{1}{2}A - 2B = 0. \end{cases}$$

This solves to $(A, B) = (4, 1)$. Hence,

$$z_1(t) = 2e^{-t} - 2e^{-6t}.$$

2. $z(0) = z(1) = c$ gives the system of linear equations

$$\begin{cases} \frac{1}{2}A - 2B = c, \\ \frac{1}{2e}A - \frac{2}{e^6}B = c, \end{cases} \implies \begin{cases} A - 4B = 2c, \\ e^5 A - 4B = 2e^6 c. \end{cases}$$

Hence,

$$A = \frac{2c(e^6 - 1)}{e^5 - 1},$$

and therefore

$$\begin{aligned}B &= \frac{A-2c}{4}\\ &= \frac{\frac{2c(e^6-1)}{e^5-1}-2c}{4}\\ &= \frac{c}{2}\cdot\frac{(e^6-1)-(e^5-1)}{e^5-1}\\ &= \frac{ce^5(e-1)}{2(e^5-1)}.\end{aligned}$$

This gives

$$z_2(t)=\frac{c(e^6-1)}{e^5-1}e^{-t}-\frac{ce^5(e-1)}{e^5-1}e^{-6t}.$$

3. Notice that

$$\begin{aligned}&\sum_{n=-\infty}^{0} z_1(t-n)\\ &= \sum_{n=-\infty}^{0}[2e^{-t+n}-2e^{-6t+6n}]\\ &= 2\sum_{n=0}^{\infty}[e^{-t-n}-e^{-6t-6n}]\\ &= 2\left[e^{-t}\sum_{n=0}^{\infty}e^{-n}-e^{-6t}\sum_{n=0}^{\infty}e^{-6n}\right]\\ &= 2\left[\frac{e^{-t}}{1-e^{-1}}-\frac{e^{-6t}}{1-e^{-6}}\right]\\ &= \frac{2e}{e-1}e^{-t}-\frac{2e^6}{e^6-1}e^{-6t}.\end{aligned}$$

Hence, c must be such that

$$\begin{cases}\dfrac{c(e^6-1)}{e^5-1}=\dfrac{2e}{e-1},\\ \dfrac{2e^6}{e^6-1}=\dfrac{ce^5(e-1)}{e^5-1}.\end{cases}$$

Both solves to precisely

$$c=\frac{2e(e^5-1)}{(e-1)(e^6-1)},$$

and hence

$$z_2(t)=\sum_{n=-\infty}^{0} z_1(t-n)$$

for this value of c.

2012.3 Question 8

1. We aim to show that for all $n \geq 0$,

$$F_n F_{n+3} - F_{n+1} F_{n+2} = F_{n+2} F_{n+5} - F_{n+3} F_{n+4}.$$

Notice that

$$\begin{aligned}
\text{RHS} &= F_{n+2} F_{n+5} - F_{n+3} F_{n+4} \\
&= F_{n+2}(F_{n+3} + F_{n+4}) - F_{n+3}(F_{n+2} + F_{n+3}) \\
&= F_{n+2} F_{n+4} - F_{n+3} F_{n+3} \\
&= F_{n+2}(F_{n+2} + F_{n+3}) - F_{n+3}(F_{n+1} + F_{n+2}) \\
&= F_{n+2} F_{n+2} - F_{n+3} F_{n+1} \\
&= F_{n+2}(F_{n+3} - F_{n+1}) - F_{n+3}(F_{n+2} - F_n) \\
&= F_n F_{n+3} - F_{n+1} F_{n+2} \\
&= \text{LHS}
\end{aligned}$$

and set $n = 0$ shows exactly what is desired.

2. By the lemma in the previous part, the problem reduces to two cases are when n is odd and when n is even.

- When n is even,

$$F_n F_{n+3} - F_{n+1} F_{n+2} = F_0 F_3 - F_1 F_2 = 0 \cdot 2 - 1 \cdot 1 = -1.$$

- When n is odd,

$$F_n F_{n+3} - F_{n+1} F_{n+2} = F_1 F_4 - F_2 F_3 = 1 \cdot 3 - 1 \cdot 2 = 1.$$

3. Using the tangent formula for sum of angles, we have

$$\begin{aligned}
\arctan\left(\frac{1}{F_{2r+1}}\right) + \arctan\left(\frac{1}{F_{2r+2}}\right) &= \arctan\left(\frac{\frac{1}{F_{2r+1}} + \frac{1}{F_{2r+2}}}{1 - \frac{1}{F_{2r+1}} \cdot \frac{1}{F_{2r+2}}}\right) \\
&= \arctan\left(\frac{F_{2r+1} + F_{2r+2}}{F_{2r+1} F_{2r+2} - 1}\right) \\
&= \arctan\left(\frac{F_{2r+3}}{F_{2r+1} F_{2r+2} + (F_{2r} F_{2r+3} - F_{2r+1} F_{2r+2})}\right) \\
&= \arctan\left(\frac{F_{2r+3}}{F_{2r} F_{2r+3}}\right) \\
&= \arctan\left(\frac{1}{F_{2r}}\right),
\end{aligned}$$

as desired.

Hence, we have

$$\arctan\left(\frac{1}{F_{2r+1}}\right) = \arctan\left(\frac{1}{F_{2r}}\right) - \arctan\left(\frac{1}{F_{2r+2}}\right),$$

and therefore

$$\begin{aligned}
\sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r+1}}\right) &= \sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r}}\right) - \sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r+2}}\right) \\
&= \sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r}}\right) - \sum_{r=2}^{\infty} \arctan\left(\frac{1}{F_{2r}}\right) \\
&= \arctan\left(\frac{1}{F_{2\cdot 1}}\right) \\
&= \arctan\left(\frac{1}{F_2}\right) \\
&= \arctan(1) \\
&= \frac{\pi}{4}.
\end{aligned}$$

2012.3 Question 12

1. Let $[S]$ denote the area (2-D case) or the volume (3-D case) of S.

   Let $l = AB = BC = CA$, and hence we have

$$[\Delta ABC] = \frac{l \cdot 1}{2} = \frac{l}{2}.$$

   By trigonometry, we also have

$$[\Delta ABC] = \frac{l^2 \sin \frac{\pi}{3}}{2} = \frac{\sqrt{3}}{4} l^2,$$

   and hence

$$\frac{\sqrt{3}}{4} l^2 = \frac{l}{2} \iff l = \frac{2}{\sqrt{3}}.$$

   On the other hand, splitting up the triangle, we have

$$\begin{aligned}[\Delta ABC] &= [\Delta ABP] + [\Delta BCP] + [\Delta ACP] \\ &= \frac{AB \cdot x_1}{2} + \frac{BC \cdot x_2}{2} + \frac{AC \cdot x_3}{2} \\ &= \frac{l}{2}\left(x_1 + x_2 + x_3\right).\end{aligned}$$

   Since $[\Delta ABC] = [\Delta ABC]$, we must have $x_1 + x_2 + x_3 = 1$.

   Let the angle bisectors of $\angle BAC, \angle ABC$ and $\angle ACB$ meet at a point O (this point exists since triangle ABC is equilateral).

   For $X_1 = \min(X_1, X_2, X_3)$, this happens if and only if P is closer to AB than BC (including the equal case, $X_1 \leq X_2$), and P is closer to AB than AC (including the equal case, $X_1 \leq X_3$). This means P must lie on the side containing point A relative to BO (inclusive), and on the side containing point B relative to AO (inclusive).

   Hence, P must lie on or inside triangle AOB, as shown in the diagram below.

   Without loss of generality (since a triangle has order-3 rotational symmetry, and the centre of symmetry is O), we only consider the case where

$$X = X_1 = \min(X_1, X_2, X_3).$$

   This means P must lie on or inside triangle AOB. Consider the cumulative distribution function of X_1 under this condition. By the following diagram, for $0 \leq x \leq \frac{1}{3}$, we must have

$$\begin{aligned}F(x) &= \mathrm{P}(X \leq x) \\ &\propto [\Delta ABO] - [\Delta ARQ] \\ &= \frac{l \cdot \frac{1}{3}}{2} \cdot \left[1 - \left(\frac{\frac{1}{3} - x}{\frac{1}{3}}\right)^2\right] \\ &= \frac{\frac{2}{\sqrt{3}} \cdot \frac{1}{3}}{2} \cdot \left[1 - (1 - 3x)^2\right] \\ &= \frac{1}{3\sqrt{3}} \cdot \left[6x - 9x^2\right] \\ &= \frac{2x - 3x^2}{\sqrt{3}}.\end{aligned}$$

   The maximum of x is $\frac{1}{3}$, and hence $F\left(\frac{1}{3}\right) = 1$. This means the constant of proportionality, k, must satisfy

$$k = \frac{F\left(\frac{1}{3}\right)}{\left[\frac{2x - 3x^2}{\sqrt{3}}\right]_{x = \frac{1}{3}}} = \frac{1}{\frac{1}{3\sqrt{3}}} = 3\sqrt{3},$$

and hence

$$F(x) = 3(2x - 3x^2).$$

Therefore, the probability density function of X for $0 \le x \le \frac{1}{3}$ must satisfy

$$f(x) = 6 - 18x = 6(1 - 3x),$$

and 0 everywhere else, i.e.

$$f(x) = \begin{cases} 6(1-3x), & 0 \le x \le \frac{1}{3}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the expectation of X satisfies

$$\begin{aligned} \mathrm{E}(X) &= \int_{\mathbb{R}} x f(x) \, \mathrm{d}x \\ &= \int_0^{\frac{1}{3}} (6x - 18x^2) \, \mathrm{d}x \\ &= \left[3x^2 - 6x^3\right]_0^{\frac{1}{3}} \\ &= 3 \cdot \left(\frac{1}{3}\right)^2 - 6 \cdot \left(\frac{1}{3}\right)^3 \\ &= \frac{3}{9} - \frac{2}{9} \\ &= \frac{1}{9}. \end{aligned}$$

2. Let the regular tetrahedron be $ABCD$ and the centroid be O. Let $AB = BC = BD = DA = l$. By trigonometry, we have

$$\frac{l^3}{6\sqrt{2}} = \frac{1}{3} \cdot \frac{\sqrt{3}l^2}{4} \cdot 1,$$

and hence

$$l = \frac{\sqrt{3}}{\sqrt{2}}.$$

Let the perpendicular distances from P to the face BCD, ACD, ABD and ABC be Y_1, Y_2, Y_3 and Y_4 respectively, and let

$$Y = \min(Y_1, Y_2, Y_3, Y_4).$$

By similar arguments as before, $Y_1 = \min(Y_1, Y_2, Y_3, Y_4)$ if and only if P is on or inside the tetrahedron $BCDO$.

Let G be the cumulative distribution function of Y_1 under this condition. For $0 \le y \le \frac{1}{4}$, we have

$$\begin{aligned} G(y) &= \mathrm{P}(Y \le y) \\ &\propto [BCDO] \cdot \left[1 - \left(\frac{\frac{1}{4} - y}{\frac{1}{4}}\right)^3\right] \\ &= \frac{1}{3} \cdot \frac{\sqrt{3}l^2}{4} \cdot \frac{1}{4} \cdot \left[1 - (1 - 4y)^3\right] \\ &= \frac{1}{16\sqrt{3}} \cdot \frac{3}{2} \cdot \left[12y - 48y^2 + 64y^3\right] \\ &= \frac{\sqrt{3}}{32} \cdot \left[12y - 48y^2 + 64y^3\right] \\ &= \frac{\sqrt{3}\left(3y - 12y^2 + 16y^3\right)}{8}. \end{aligned}$$

Since the maximum of y is $\frac{1}{4}$, we must have $G\left(\frac{1}{4}\right) = 1$, and hence the constant of proportionality, k, must satisfy

$$k = \frac{G\left(\frac{1}{4}\right)}{\left[\frac{\sqrt{3}(3y - 12y^2 + 16y^3)}{8}\right]_{y = \frac{1}{4}}} = \frac{1}{\frac{\sqrt{3}}{32}} = \frac{32}{\sqrt{3}}.$$

Hence,

$$G(y) = 4\left(3y - 12y^2 + 16y^3\right),$$

and the probability density function of Y must satisfy for $0 \leq y \leq \frac{1}{4}$

$$g(y) = 4\left(3 - 24y + 48y^2\right) = 12\left(1 - 8y + 16y^2\right).$$

Hence,

$$\begin{aligned}
\mathrm{E}(y) &= \int_{\mathbb{R}} y g(y)\,\mathrm{d}y \\
&= \int_0^{\frac{1}{4}} 12\left(y - 8y^2 + 16y^3\right)\mathrm{d}y \\
&= \left[6y^2 - 32y^3 + 48y^4\right]_0^{\frac{1}{4}} \\
&= 6 \cdot \left(\frac{1}{4}\right)^2 - 32 \cdot \left(\frac{1}{4}\right)^3 + 48 \cdot \left(\frac{1}{4}\right)^4 \\
&= \frac{3}{8} - \frac{1}{2} + \frac{3}{16} \\
&= \frac{3 \cdot 2 - 1 \cdot 8 + 3}{16} \\
&= \frac{1}{16}.
\end{aligned}$$

2012.3 Question 13

1. We have

$$\begin{aligned}
\mathrm{E}(Z \mid a < Z < b) &= \frac{\int_a^b z\Phi'(z)\,\mathrm{d}z}{\int_a^b \Phi'(z)\,\mathrm{d}z} \\
&= \frac{\int_a^b ze^{-\frac{z^2}{2}}\,\mathrm{d}z}{\sqrt{2\pi}\left(\Phi(b) - \Phi(a)\right)} \\
&= \frac{\left[-e^{-\frac{z^2}{2}}\right]_a^b}{\sqrt{2\pi}\left(\Phi(b) - \Phi(a)\right)} \\
&= \frac{e^{-\frac{a^2}{2}} - e^{-\frac{b^2}{2}}}{\sqrt{2\pi}\left(\Phi(b) - \Phi(a)\right)}
\end{aligned}$$

2. Since $X = \mu + \sigma Z$

$$\begin{aligned}
\mathrm{E}(X \mid X > 0) &= \mathrm{E}(\mu + \sigma Z \mid (\mu + \sigma Z) > 0) \\
&= \mu + \sigma\,\mathrm{E}(Z \mid (\mu + \sigma Z) > 0) \\
&= \mu + \sigma\,\mathrm{E}\left(Z \mid Z > -\frac{\mu}{\sigma}\right),
\end{aligned}$$

as desired.

Hence,

$$\begin{aligned}
m &= \mathrm{E}(|X|) \\
&= \mathrm{E}(|X| \mid X > 0) \cdot \mathrm{P}(X > 0) + \mathrm{E}(|X| \mid X < 0) \cdot \mathrm{P}(X < 0) \\
&= \mathrm{E}(X \mid X > 0) \cdot \mathrm{P}(X > 0) - \mathrm{E}(X \mid X < 0) \cdot \mathrm{P}(X < 0) \\
&= \left[\mu + \sigma\,\mathrm{E}\left(Z \mid Z > -\frac{\mu}{\sigma}\right)\right] \cdot \mathrm{P}(\mu + \sigma Z > 0) \\
&\quad - \left[\mu + \sigma\,\mathrm{E}\left(Z \mid Z < -\frac{\mu}{\sigma}\right)\right] \cdot \mathrm{P}(\mu + \sigma Z < 0) \\
&= \left[\mu + \sigma \cdot \frac{\exp\left(-\frac{1}{2}\left(-\frac{\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}\left(1 - \Phi\left(-\frac{\mu}{\sigma}\right)\right)}\right] \cdot \left[1 - \Phi\left(-\frac{\mu}{\sigma}\right)\right] - \left[\mu + \sigma \cdot \frac{-\exp\left(-\frac{1}{2}\left(-\frac{\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}\Phi\left(-\frac{\mu}{\sigma}\right)}\right] \cdot \Phi\left(-\frac{\mu}{\sigma}\right) \\
&= \mu\left[1 - \Phi\left(-\frac{\mu}{\sigma}\right) - \Phi\left(-\frac{\mu}{\sigma}\right)\right] + \frac{\sigma\exp\left(-\frac{1}{2}\left(-\frac{\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}} \cdot (1 + 1) \\
&= \mu\left[1 - 2\Phi\left(-\frac{\mu}{\sigma}\right)\right] + \frac{\sqrt{2}\sigma\exp\left(-\frac{1}{2} \cdot \frac{\mu^2}{\sigma^2}\right)}{\sqrt{\pi}},
\end{aligned}$$

as desired.

To find the variance of $|X|$, we would like to find $\mathrm{E}(|X|^2)$. But this is precisely $\mathrm{E}(|X|^2) = \mathrm{E}(X^2) = \mathrm{Var}(X) + \mathrm{E}(X)^2 = \sigma^2 + \mu^2$. Hence,

$$\begin{aligned}
\mathrm{Var}(|X|) &= \mathrm{E}(|X|^2) - \mathrm{E}(|X|)^2 \\
&= \sigma^2 + \mu^2 - m^2.
\end{aligned}$$